AEDC-TR-76-163

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SMOOTHING TECHNIQUE AND VARIANCE PROPAGATION FOR ABEL INVERSION OF SCATTERED DATA

ENGINE TEST FACILITY
ARNOLD ENGINEERING DEVELOPMENT CENTER
AIR FORCE SYSTEMS COMMAND
ARNOLD AIR FORCE STATION, TENNESSEE 37389

April 1977

Final Report for Period January 1975 — June 1976

Prepared for

DIRECTORATE OF TECHNOLOGY
ARNOLD ENGINEERING DEVELOPMENT CENTER
ARNOLD AIR FORCE STATION, TENNESSEE 37389

NOTICES

APPROVAL STATEMENT

This technical report has been reviewed and is approved for publication.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1 REPORT NUMBER AEDC-TR-76-163	2 GOVT ACCESSION NO.	3 RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SMOOTHING TECHNIQUE AND VARIANCE PROPAGATION FOR ABEL INVERSION OF SCATTERED DATA		5 TYPE OF REPORT & PERIOD COVERED Final Report - January 1975 - June 1976
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) R. T. Shelby, University of Tennessee Space Institute and C. C. Limbaugh, ARO, Inc.		8. CONTRACT OR GRANT NUMBER(s)
9 PERFORMING ORGANIZATION NAME AND ADDRESS Arnold Engineering Development Center (DY) Air Force Systems Command Arnold Air Force Station, Tennessee 37389		10 PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program Element 65807F
11. CONTROLLING OFFICE NAME AND ADDRESS Arnold Engineering Development Center (DYFS) Arnold Air Force Station, Tennessee 37389		12 REPORT DATE April 1977
		13. NUMBER OF PAGES 103
14 MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A

16 DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18 SUPPLEMENTARY NOTES

Available in DDC

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

errors
propagation
analysis
least squares method
polynomials
radiation

20 ABSTRACT (Continue on reverse side if necessary and identify by block number)

An analytic method of performing Abel inversions and subsequent analysis of the propagated experimental errors is described. The Abel inversion is applied to the problem of determining the radial distribution of emission coefficients from measurements of the radiance from a cylindrically symmetric radiating source. The particular scheme investigated is a least-squares polynomial spline fit technique. The spline fit technique involves modeling of the raw data by a series of polynomials, with each polynomial

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20. ABSTRACT (Continued)

applying over a different domain of the data. The polynomials are constrained so that the total data profile is smooth and yet provides the best fit of all the data in the least-squares sense. The resultant polynomial model of the data is inverted analytically. The associated error propagation analysis is developed by casting the numerical equations selected to perform the curve fit and integration into a form whereby the problem can be viewed as a linear transformation from the raw data to the inverted results. In this manner, the variance-covariance matrix of the raw data can be directly transformed to the variance-covariance matrix of the emission coefficients; therefore, the standard deviations of the intensity data points can be related directly to the standard deviations of the resultant emission coefficients. The result is an objective method of determining which series of polynomials yields the best fit to the raw data and the best inversion results.

PREFACE

The work reported herein was conducted by the Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC) under Program Element 65807F. The results of the research were obtained by ARO, Inc., AEDC Division (a Sverdrup Corporation Company), operating contractor for the AEDC, AFSC, Arnold Air Force Station, Tennessee, under ARO Project Numbers R33A-00A and R32S-06A. The authors of this report were R. T. Shelby, University of Tennessee Space Institute, and Dr. C. C. Limbaugh, ARO, Inc. The manuscript (ARO Control No. ARO-ETF-TR-76-126) was submitted for publication on October 27, 1976.

The authors wish to express special appreciation to Mr. F. C. Loper, ARO, Inc., whose original work with the least-squares spline fitting technique at AEDC provided the inspiration for the approach used here.

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1.0 INTRODUCTION

1.1 BACKGROUND AND OBJECTIVE

A problem which has long been of importance to the experimentalist is the determination of radially dependent physical properties in axisymmetric sources from measurements of integrated properties along a line of sight. One important example of this type of problem is the calculation of the radial distribution of emission coefficients from observations of the radiance (radiated power per unit area per unit solid angle) from a cylindrically symmetric radiating source. When the source is optically thin, the solution to a problem of this type is usually reached by the use of the Abel transform (Ref. 1). The application of the Abel transform in this instance requires the emission coefficients to have cylindrical symmetry. With the cylindrical symmetry assumption, the measured radiance, I(x), can be written (Fig. 1) as

$$I(x) = 2 \int_x^R \frac{\epsilon(r)\, r\, dr}{(r^2 - x^2)^{1/2}} \tag{1}$$

Here I(x) is the measured radiance (radiated power per unit area per unit solid angle), which is a function of the displacement x, r is the local radius, R is the maximum radius of the source, and $\varepsilon(r)$ is the emission coefficient (radiated power per unit volume per unit solid angle). Note that I at each x is the result of integrating the emission coefficient ε across the extent of the source. The factor 2 arises because of the cylindrical symmetry assumption. Equation (1) is a form of the Abel integral equation (Ref. 2).

The use of Abel's transformation yields

$$\epsilon(r) = -\frac{1}{\pi} \int_r^R \frac{dI/dx}{(x^2 - r^2)^{1/2}}\, dx \tag{2}$$

Equation (2) is the inversion equation which gives the radial emission coefficient in terms of the geometry of the source and the measured radiance distribution. Details of the development of Eqs. (1) and (2) are found, for example, in Ref. 1.

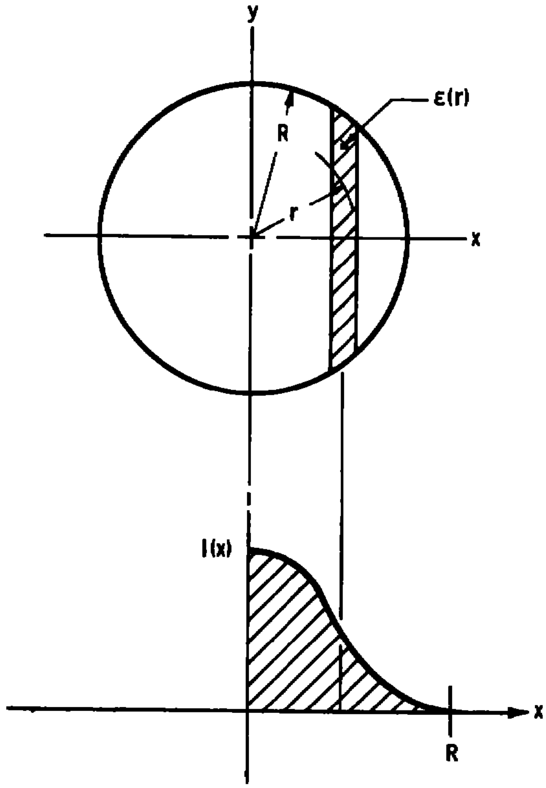

Figure 1. Geometry of the axisymmetric source.

A number of methods have been applied to the solution of Eqs. (1) and (2) to obtain the radial distribution of the unknown emission coefficient ε(r). An early approach documented by Pearce (Ref. 3) depends upon finding the areas of homogeneous zones in the plasma and replacing the integral by a summation over the zones. Nestor and Olsen (Ref. 4) make a transformation of variables such that $r^2 = v$ and $x^2 = u$ and assume that $I'(u)$ is constant over each small interval. The subsequent series of integrals may then be evaluated, and Eq. (2) is approximated by a sum. Bockasten (Ref. 5) fitted third-degree polynomials exactly to the

data points and approximated Eq. (2) with a sum. Numerical methods such as those described in Refs. 3 through 5, which do not use smoothing of the raw data, have an intrinsic disadvantage in that small errors in the radiance can lead to fairly large errors in the emission coefficient because of inaccuracies in the derivative of I(x).

Some investigators have used smoothing techniques utilizing curve fitting or other mathematical approximations in an attempt to reduce the effect of experimental error in the data. Freeman and Katz (Ref. 6) least squares fitted a single curve to the raw data. Barr (Ref. 7) used least-squares polynomials to determine the best fit of the data over a number of intervals centered about each data point. Birkebak and Cremers (Ref. 8) used a method similar to that of Barr. The data were least-squares curve fitted to polynomials over a number of data intervals. Dooley and McGregor (Ref. 9) directly applied the integral in Eq. (2) by using the experimental data and a coordinate transformation of the integrand to provide a means of numerically solving the integral.

Comparisons of various techniques (Ref. 8) indicated that, in general, smoothing techniques yield better final results than other methods, particularly when there is appreciable scatter in the data.

A major problem with any method of solving the problem described by Eq. (2) is that of determining the effect of experimental error in the measured radiance values upon the resultant values of the emission coefficient. In an effort to obtain error propagation information, some investigators have applied their methods of solution to simple problems in which analytic solutions could be determined (Refs. 8 and 10). By using scattered and unscattered input data (i.e., radiance values) and by comparing the resultant emission coefficient values with their analytic values, the investigators were able to obtain empirical information on the error propagation characteristics of the techniques as applied to particular test problems. However, at present, no general

analytic method has been developed which, after a prior analysis of the input data, follows the propagation of experimental error throughout the numerical steps of the specific technique. It is the purpose of this investigation to describe a method for smoothing the data for inversion and to develop a concomitant error propagation analysis.

1.2 CRITERIA FOR CURVE FIT CHOICES

Before proceeding to the description of the specific technique, it would be useful to consider some of the more subtle consequences of curve fitting in order to define a criterion by which the curve fit choices used in the work reported herein were made.

Experimental data are subject to random uncertainties and are generated by a physical phenomenon for which one does not necessarily know the functional form. Indeed, specification of the functional form, or an "adequate" approximation thereof, is generally the objective of the analysis. Because of the experimental scatter in the data, one may be uncertain of its correct functional form. The usual approach in such situations is to curve fit the data to some smooth functional form either by analytical or graphical techniques. When the analytic approach is used, the data are often fit to a specific polynomial in the independent variable. However, as is often readily apparent, the first choice of polynomial often cannot do an "adequate" job of fitting the data, and other polynomials are tried. This failure of fit may be caused by the fact that the data are simply not expressible by a polynomial function. Similarly, for the same reason, there are often several choices for polynomials which appear to "adequately" fit the data. Yet, each polynomial is different and will yield somewhat different results. Hence the results of a smoothing process cannot be considered unique insofar as the underlying physical phenomena are concerned, and there can be a large number of possible solutions. (Note that, for a given set of data and a specific polynomial form, the numerical solution is unique; it is the resultant, derived description of the underlying physical phenomena which is not unique).

The observation that it is generally unreasonable to expect a polynomial to provide an adequate representation of some physical phenomenon over a wide range of the independent variable prompted the approach described herein. The data are divided into several intervals, and a different polynomial is assumed valid over each interval. Since physical phenomena are generally smooth, the polynomials are constrained to be smooth and continuous at interval boundaries. Since the data are expected to be randomly scattered, a final least squares constraint is imposed to provide smoothing capabilities.

If the experimental error is propagated through a curve fitting process, one can obtain an estimate for the error, or uncertainty in the results. Even though the correct functional form may not be that chosen for fitting, the errors are propagated as though the chosen function is the correct one. If the chosen function provides an "adequate" representation of the raw data, it is expected that the derived results provide an "adequate" representation of the true physical phenomenon. Thus the propagated errors induced by the data uncertainties are an "adequate" representation of the uncertainties in interpretation of the physical phenomenon. For the application described herein, when several functions appear to give "adequate" curve fits, their results and propagated uncertainties agreed well with each other. This observation makes a strong favorable argument that the results are in fact descriptive of the underlying physical phenomenon. However, it must be remembered that the propagated uncertainties describe the uncertainties in the data fitting the chosen form and not the uncertainties in the chosen function fitting the physical phenomenon.

Since, for a given set of data, there may be a wide choice of possible curve fits, the choice of which curve fit to use is highly subjective and generally must be left to the investigator. The criteria for determining which result to use are as follows:

1. Choose the set of functions which performs an acceptable fit to the data, based on the observed error propagation characteristics.

2. From the set of functions chosen by (1), pick the function which provides the best error propagation characteristics for the problem.

Although the foregoing discussion may appear obvious to the experienced data analyst, the underlying theme has been that there can be no procedure or set of conditions which will unequivocably define the correct curve fits in all cases. The error propagation analysis provides only additional objective data for what is finally a subjective decision.

1.3 TECHNICAL APPROACH

The numerical approach is based on a curve-fitting method which utilizes a least-squares polynomial spline fit technique (Ref. 11) to smooth the data. The Abel inversion is effected by dividing the raw data into several small intervals and obtaining a least-squares polynomial curve fit to the data in each interval. Since the integral is a linear operator, the emission coefficient integral, Eq. (2), is expressed as the sum of several integrals; each integral applies over a different interval of data, and in each case the radiance data are expressed by a different polynomial. The emission coefficient is expressed as

$$\epsilon(r) = -\frac{1}{\pi} \int_{r}^{Z_{j-1}} \frac{dI_{j-1}/dx}{(x^2 - r^2)^{1/2}}\, dx - \frac{1}{\pi} \sum_{i=j}^{n} \int_{Z_{j-1}}^{Z_j} \frac{dI_i/dx}{(x^2 - r^2)^{1/2}}\, dx \qquad (3)$$

where the subscript i identifies the particular interval for evaluation, Z_{i-1} and Z_i denote the endpoints of the interval, n is the number of intervals required to span the data, and in all cases r is less than Z_{j-1}.

For the values of the emission coefficients to be physically correct, the polynomials must be constrained to be smooth and single valued at interval boundaries and also must satisfy the assumption of cylindrical symmetry. The cylindrical symmetry assumption is easily accounted for by using an even function of the form

$$I_i(x) = a_{i1} + a_{i2}x^2 + a_{i3}x^4 + a_{i4}x^6 \tag{4}$$

where the subscript i denotes the i^{th} interval. The functions in adjacent intervals are designed so that their ordinate, slope, and second derivatives have the same value at the endpoints of adjacent intervals. Thus, the coefficients of a sixth-degree, even polynomial over each interval of data are determined such that a best fit in the least-squares sense over the entire set of data is obtained with the condition that the polynomials and their first derivatives are smooth and their second derivates are continuous at the interval boundaries. With I(x) represented by a series of polynomials in the form of Eq. (4), the integrals become expressible by direct integration in closed form over each interval of the spline fit.

Another constraint upon the functional representation of the physical data which is imposed by the development of Eq. (2) is that it must have zero slope and ordinate at the boundary of the cylinder. This constraint is imposed artifically by obtaining a new curve over the last interval after the other intervals have been fit with the proper constraints. The constraints of continuous first derivative and smooth ordinate are maintained at the interval boundary for this artifical curve. The construction of the artifical curve reduces the accuracy of the curve fit intensity values in the last interval. However, the curve fit values of all the other intervals are unaffected, and in general the data in the last interval are very near to zero, quite inaccurate, and insignificant with respect to the rest of the data. When the results of the inversion in the last interval are important, the method described herein must be applied cautiously. A computer program to perform the analysis developed herein is described in Appendix A.

2.0 NUMERICAL INVERSION TECHNIQUE AND ERROR ANALYSIS

2.1 NUMERICAL TECHNIQUE

As was indicated in Section 1.0, the problem which prompted the present investigation was the development of a method to analyze the effect of experimental error in observed radiance data upon the emission coefficient values obtained by the solution to the Abel inversion problem. A measure of the experimental error in the radiance data is provided by the standard deviations of the radiance values which are determined from the experimental data.

From statistics theory, it is known that if a matrix vector equation, A = MB, can be written where M is a transformation matrix relating the vectors A and B, then a relationship between the standard deviations of the elements of vectors A and B can be derived. The first step in the error analysis is the development of the matrix-vector equation, E = FY, where Y is a vector containing the radiance data values, and F is a matrix relating the vectors Y and E.

Since the data are to be divided into intervals, denote the displacement values which are endpoints for the k^{th} interval by Z_{k-1} and Z_k and $k = 1,\ldots, n$. Note that for n intervals there are $(n + 1)$ z_j values, $j = 0,\ldots, n$. The polynomial in each interval can be written as

$$P_k(x) = \sum_{i=1}^{4} a_{ki}\, x^{2i-2} \quad ; \quad k = 1, 2, \ldots, n \qquad Z_{k-1} < x < Z_k \tag{5}$$

where n is the number of intervals into which the data points have been separated. The coefficients a_{ki} will in general be different for the polynomials representing each interval.

Since the raw data in each interval are to be least-squares curve fit to polynomials of the form of Eq. (5), an expression for the sum of the squares of the deviations of the curve fit values from the actual (input) radiance values is needed for each interval. For this purpose, let

$$S_k\,(a_{k1}, a_{k2}, a_{k3}, a_{k4}) = \sum_{i=1}^{m_k} \Big[I(x_j) - P_k\,(x_j) \Big]^2 \;:\; k = 1, 2, \ldots, n \tag{6}$$

$$Z_{k-1} < x_j < Z_k$$

where n is the number of intervals, m_k is the number of data points in the k^{th} interval, and the displacements x_j are numbered independently in each interval. The value of S_k in each interval must now be minimized subject to the constraints:

$$\begin{aligned}
\phi_{k,1}\,(a_{k1}, a_{k2}, a_{k3}, a_{k4}) &= P_k\,(Z_k) - P_{k+1}\,(Z_k) = 0 \\
\phi_{k,2}\,(a_{k1}, a_{k2}, a_{k3}, a_{k4}) &= P_k'(Z_k) - P_{k+1}'\,(Z_k) = 0 \\
\phi_{k,3}\,(a_{k1}, a_{k2}, a_{k3}, a_{k4}) &= P_k''(Z_k) - P_{k+1}''\,(Z_k) = 0 \\
& \qquad k = 1, 2, \ldots, (n-1)
\end{aligned} \tag{7}$$

where $P_k'(x)$ is the first derivative of the function $P_k(x)$, with respect to the coordinate x, and $P_k''(x)$ is the second derivative. These constraints express the conditions required for the ordinate, slope, and second derivatives of the fitting polynomials for adjacent intervals to be continuous at the interval boundary. This continuity constrains the formulation to provide a more nearly correct representation of the real physical data.

Lagrange's method of undetermined multipliers (Refs. 12 and 13) can be used to minimize Eq. (6) subject to the constraints expressed by Eqs. (7). Let

$$F_1(a_{11}, a_{12}, a_{13}, a_{14}) = S_1 + \sum_{j=1}^{3} \lambda_{1,j}\phi_{1,j}$$

$$F_k(a_{k1}, a_{k2}, a_{k3}, a_{k4}) = S_k + \sum_{j=1}^{3} (\lambda_{k-1,j}\phi_{k-1,j} + \lambda_{kj}\phi_{kj}) \tag{8}$$

$$F_n(a_{n1}, a_{n2}, a_{n3}, a_{n4}) = S_n + \sum_{i=1}^{3} \lambda_{n-1,j}\phi_{n-1,j}$$

$$k = 2, 3, \ldots, (n-1)$$

where n is the number of intervals and the λ_{mj} values are the Lagrange multipliers.

The parameters, λ_{mj}, and the coefficients, a_{ki}, which effect the necessary minimization are determined from the 4n equations

$$(½)\frac{\partial F_k}{\partial a_{ki}} = 0; \quad \begin{array}{l} k = 1, 2, \ldots, n \\ i = 1, 2, 3, 4 \end{array} \tag{9}$$

and the 3(n-1) equations

$$(½)\phi_{m,j} = 0; \quad \begin{array}{l} m = 1, 2, \ldots (n-1) \\ j = 1, 2, 3 \end{array} \tag{10}$$

The constant multiplier, 1/2, is introduced in Eqs. (9) and (10) to put the equations in a form which is convenient for subsequent developments.

By expanding Eq. (9) for k = 1, one obtains

$$a_{11} m_1 + a_{12} \sum_{i=1}^{m_1} x_i^2 + a_{13} \sum_{i=1}^{m_1} x_i^4 + a_{14} \sum_{i=1}^{m_1} x_i^6 + \tfrac{1}{2} \lambda_{11} = \sum_{i=1}^{m_1} I_i$$

$$a_{11} \sum_{i=1}^{m_1} x_i^2 + a_{12} \sum_{i=1}^{m_1} x_i^4 + a_{13} \sum_{i=1}^{m_1} x_i^6 - a_{14} \sum_{i=1}^{m_1} x_i^8 + \frac{Z_1^2}{2} \lambda_{11} + Z_1 \lambda_{12}$$
$$+ \lambda_{13} = \sum_{i=1}^{m_1} x_i^2 I_i$$

$$a_{11} \sum_{i=1}^{m_1} x_i^4 + a_{12} \sum_{i=1}^{m_1} x_i^6 + a_{13} \sum_{i=1}^{m_1} x_i^8 + a_{14} \sum_{i=1}^{m_1} x_i^{10} + \frac{Z_1^4}{2} \lambda_{11} + 2Z_1^3 \lambda_{12} \tag{11}$$
$$+ 6Z_1^2 \lambda_{13} = \sum_{i=1}^{m_1} x_i^4 I_i$$

$$a_{11} \sum_{i=1}^{m_1} x_i^6 - a_{12} \sum_{i=1}^{m_1} x_i^8 - a_{13} \sum_{i=1}^{m_1} x_i^{10} + a_{14} \sum_{i=1}^{m_1} x_i^{12} + \frac{Z_1^6}{2} \lambda_{11}$$
$$+ 3Z_1^5 \lambda_{12} + 15Z_1^4 \lambda_{13} = \sum_{i=1}^{m_1} x_i^6 I_i$$

where the x_i's are the displacement values, Z_1 is the second endpoint, m_1 is the number of points in the first interval, and $I_i = I(x_i)$. The constant multiplier, 1/2, in Eq. (9) has the effect of eliminating the factor of 2, which is introduced in Eq. (9) by the differentiation. Expanding Eq. (9) for k = 2, 3, ..., n yields more equations similar in form to Eq. (11).

Expanding Eq. (10) for m = 1 yields

$$
\begin{aligned}
&\tfrac{1}{2}a_{11} + \frac{Z_1^2}{2}a_{12} - \frac{Z_1^4}{2}a_{13} + \frac{Z_1^6}{2}a_{14} - \tfrac{1}{2}a_{21} - \frac{Z_1^2}{2}a_{22} - \frac{Z_1^4}{2}a_{23} - \frac{Z_1^6}{2}a_{24} = 0 \\
&Z_1a_{12} + 2Z_1^3a_{13} + 3Z_1^5a_{14} - Z_1a_{22} - 2Z_1^3a_{23} - 3Z_1^5a_{24} = 0 \qquad (12) \\
&a_{12} + 6Z_1^2a_{13} + 15Z_1^4a_{14} - a_{22} - 6Z_1^2a_{23} - 15Z_1^4a_{24} = 0
\end{aligned}
$$

Equations of similar form are obtained for m = 2, 3,..., (n-1).

Equations (4) and (10), with Eqs. (11) and (12) describing representative algebraic details, are the least-squares spline fit equations and represent a system of equations which can be written in the matrix-vector notation

$$BA = C \qquad (13)$$

where B is a (7n - 3) by (7n - 3) symmetric matrix, A is a (7n - 3) by 1 matrix which has as elements the 4n coefficients, a_{ki}, and the 3n - 3 multipliers, λ_{mj}, and C is a (7n - 3) by 1 matrix, where n is the number of intervals into which the data points have been divided.

The matrix B can be partitioned into

$$B = \begin{bmatrix} R & N \\ N^T & 0 \end{bmatrix} \qquad (14)$$

where R is a 4n x 4n block diagonal matrix which is defined as

$$R = \begin{bmatrix} P_1 & & & 0 \\ & P_2 & & \\ & & \ddots & \\ 0 & & & P_n \end{bmatrix} \qquad (15)$$

The matrices P_j are 4 by 4 symmetric matrices which are defined (letting, for convenience, $\ell_j = m_1 + m_2 + \ldots + m_j$ and $S_j = m_1 + m_2 \ldots + m_{j-1} + 1$)

$$P_j = \begin{bmatrix} m_j & \sum_{i=S_j}^{\ell_j} x_i^2 & \sum_{i=S_j}^{\ell_j} x_i^4 & \sum_{i=S_j}^{\ell_j} x_i^6 \\ \sum_{i=S_j}^{\ell_j} x_i^2 & \sum_{i=S_j}^{\ell_j} x_i^4 & \sum_{i=S_j}^{\ell_j} x_i^6 & \sum_{i=S_j}^{\ell_j} x_i^8 \\ \sum_{i=S_j}^{\ell_j} x_i^4 & \sum_{i=S_j}^{\ell_j} x_i^6 & \sum_{i=S_j}^{\ell_j} x_i^8 & \sum_{i=S_j}^{\ell_j} x_i^{10} \\ \sum_{i=S_j}^{\ell_j} x_i^6 & \sum_{i=S_j}^{\ell_j} x_i^8 & \sum_{i=S_j}^{\ell_j} x_i^{10} & \sum_{i=S_j}^{\ell_j} x_i^{12} \end{bmatrix} \tag{16}$$

N is a 4n by (3n - 3) block band matrix defined as

$$N = \begin{bmatrix} Q_1 & & & \\ -Q_1 & Q_2 & & \\ & -Q_2 & \ddots & \\ & & \ddots & Q_{n-1} \\ & & & -Q_{n-1} \end{bmatrix} \tag{17}$$

The matrix N consists of a block diagonal and one block lower co-diagonal. N^T is the transpose of the matrix N. The matrices Q_j are 4 by 3 matrices defined as

$$Q_j = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{Z_j^2}{2} & Z_j & 1 \\ \frac{Z_j^4}{2} & 2Z_j^3 & 6Z_j^2 \\ \frac{Z_j^6}{2} & 3Z_j^5 & 15Z_j^4 \end{bmatrix} \tag{18}$$

Noting the right side of Eqs. (11) and (12), one can write the matrix C as

$$C = GY \tag{19}$$

where G is a (7n - 3) by p matrix of certain powers of x, the p-independent data points, with the elements of the last 3n - 3 rows all zero, and y is a p by 1 matrix of the p-dependent variable data points I(x) (that is, the radiance values). Note that $p = m_1 + m_2 + \ldots + m_n$.

For ease of representation the matrix G may be partitioned into a block diagonal matrix

$$G = \begin{bmatrix} S_1 & & & \\ & S_2 & & 0 \\ & & \ddots & \\ & 0 & & S_n \end{bmatrix} \tag{20}$$

where each S_j is a 4 by m_j matrix:

$$S_j = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x^2_{m_{j-1}-1} & x^2_{m_{j-1}-2} & \cdots & x^2_{m_{j-1}+m_j} \\ x^4_{m_{j-1}+1} & x^4_{m_{j-1}+2} & \cdots & x^4_{m_{j-1}+m_j} \\ x^6_{m_{j-1}+1} & x^6_{m_{j-1}-2} & \cdots & x^6_{m_{j-1}+m_j} \end{bmatrix} \tag{21}$$

Writing Eq. (13) as

$$A = B^{-1} C \tag{22}$$

and substituting Eq. (19) yields

$$A = B^{-1} G Y \tag{23}$$

where B^{-1} is a (7n - 3) by (7n - 3) matrix, which is the inverse of B. Equation (23) can be written as

$$A = W Y \tag{24}$$

where A is the (7n - 3) by 1 matrix, which has as elements the coefficients, a_{ki}, and the multipliers, λ_{mj}. Y is the p by 1 matrix of data intensity values, and W equals $B^{-1}G$, a (7n - 3) by p matrix.

Performing the integration of Eq. (3) with the polynomials of the form of Eq. (4) yields

$$\begin{aligned} \epsilon(r) = & -\frac{(x^2 - r^2)^{1/2}}{\pi} \left[2a_{2,j-1} + \frac{4}{3} a_{3,j-1} (x^2 + 2r^2) + \frac{6}{15} a_{4,j-1} \right. \\ & \left. \cdot (3x^4 + 4x^2r^2 + 8r^4) \right] \Bigg|_r^{Z_{i-1}} - \sum_{i=j}^{n} \frac{(x^2 - r^2)^{1/2}}{\pi} \left[2a_{2i} \right. \\ & \left. + \frac{4}{3} a_{3i} (x^2 + 2r^2) + \frac{6}{15} a_{4i} (3x^4 + 4x^2r^2 + 8r^4) \right] \Bigg|_{Z_{i-1}}^{Z_i} \end{aligned} \tag{25}$$

where k is chosen so that r is always less than Z_k. Expanding Eq. (25) for several values of r, one can write the resulting system of equations in the matrix-vector form

$$E = MA \tag{26}$$

where E is an m by 1 matrix of values of the emission coefficient $\varepsilon(r)$ evaluated at m different values of r, A is a (7n -3) by 1 matrix defined by Eq. (13), and M is an m by (7n - 3) matrix with the elements of the last 3n - 3 columns all zero.

Substituting into Eq. (26), the expression for the matrix A from Eq. (24) yields

$$E = M\mathbb{W}Y \tag{27}$$

By performing the matrix multiplication

$$F = M\mathbb{W} \tag{28}$$

Eq. (22) can be written as

$$E = FY \tag{29}$$

where E is the m by 1 matrix of values of the emission coefficient $\varepsilon(r)$ evaluated at m different values of r, Y is a p by 1 matrix of the p-dependent variable data points I(x), and F is an m by p matrix providing the transformation from the Y space to the E space.

2.2 ERROR ANALYSIS TECHNIQUE

To complete the development of the technique to include the error propagation analysis, it is necessary to utilize some definitions and results from statistics theory (Refs. 14 and 15). The expected value of a random variable x, denoted by $\mathcal{E}(x)$, is obtained by finding the average value of the function over all possible values of the variable. The expected value of a matrix or vector M, denoted by $\mathcal{E}(M)$, is the matrix of the expected values of the elements of M. The moments of a distribution are the expected values of the powers of the variable which have the given distribution.

Let μ_ε denote the first moment or mean of the emission coefficient function $\varepsilon(r)$ and let μ_I denote the mean of the radiance function $I(x)$. Then

$$\mu_\epsilon = \mathcal{E}(E) = \mathcal{E}(FY) = F\mathcal{E}(Y) = F\mu_I \tag{30}$$

The covariance matrix of E, denoted by $[E]_{cv}$, is the m by m symmetric matrix defined as

$$[E]_{cv} = \mathcal{E}\left[(E-\mu_\epsilon)(E-\mu_\epsilon)^T\right] \tag{31}$$

where m is the number of different values of r at which the emission coefficient $\varepsilon(r)$ is evaluated. The following steps yield an expression which can be used to evaluate the covariance matrix of E:

$$\begin{aligned} [E]_{cv} &= \mathcal{E}\left[(E-\mu_\epsilon)(E-\mu_\epsilon)^T\right] \\ &= \mathcal{E}\left[(FY - F\mu_I)(FY - F\mu_I)^T\right] \\ &= \mathcal{E}\left[F(Y-\mu_I)(F(Y-\mu_I))^T\right] \\ &= \mathcal{E}\left[F(Y-\mu_I)(Y-\mu_I)^T F^T\right] \\ &= F\left(\mathcal{E}\left[(Y-\mu_I)(Y-\mu_I)^T\right]\right)F^T \\ [E]_{cv} &= F[Y]_{cv}F^T \end{aligned} \tag{32}$$

The term $[Y]_{cv}$ in Eq. (32) denotes the covariance matrix of Y, which is an m by m symmetric matrix.

The diagonal elements of the matrix $[Y]_{cv}$ are identified with the squares of the standard deviations of the radiance measurements. The nondiagonal elements are the covariances between the various elements of the matrix Y. In the present problem, since the individual observations are independent, then the elements of Y (that is, the radiance data) are assumed to be uncorrelated. Therefore, all of the nondiagonal elements of the matrix $[Y]_{cv}$ are zero, and thus $[Y]_{cv}$ is simply a diagonal matrix containing the variances of the radiance data.

The matrix F provides the transformation from the Y space to the E space. The form of the matrix is determined by the raw data and the choice of numerical technique. The $[E]_{cv}$ is the covariance matrix of E, and the diagonal elements are the variances of the emission coefficient values which have been calculated by the least-squares spline fit approximation to the data. The off-diagonal elements of the matrix $[E]_{cv}$ give the covariance between the various emission coefficient values. The standard deviations of the emission coefficient values are the positive square roots of the diagonal elements of $[E]_{cv}$. Thus, determination of the matrix F, Eq. (28), and its use in Eqs. (29) and (32) provides the formal solution to the inversion problem and the propagation of the random errors associated with the measurements.

3.0 RESULTS AND DISCUSSION

3.1 INTRODUCTION

In the determination of the solution to a specific inversion problem, described formally by Eqs. (27) and (30), there are generally four parameters which may be varied by the user. These four parameters are (1) the total number of data points, (2) the distribution of the data points chosen for analysis, (3) the number of intervals into which the data are divided, and (4) the number of points in each interval. Variation of each of these parameters affects the elements of the matrix F, Eq. (28), and the subsequent inversion and random error propagation results. Included in this section are results of the Abel inversion of several sets of analytic data chosen to illustrate the effect of variations of the above four parameters on the results of the inversion. Also included, as an illustration of the application of the method to typical experimental data, are the results of the inversion of data taken in a recent research experiment.

3.2 ANALYTIC CONTINUOUS TEST DATA

Thirty-one data points are used to illustrate the technique. These data points were generated from the function

$$I(x) = e^{-x^2} \; ; \quad 0 \leq x \leq 3.0 \tag{33}$$

which may be inverted by Eq. (2) to yield the analytic function

$$\epsilon(r) = \frac{1}{\sqrt{\pi}} \; e^{-r^2} \, \mathrm{erf}\left[(R^2 - r^2)^{1/2}\right] \tag{34}$$

The number of data points and the displacement distribution were fixed, leaving only the number of intervals and the number of points per interval to be varied. The inversion was applied to several cases with different values of the free parameters. A standard deviation of 10 percent of each radiance value was assumed in each case. The displacement radiance and standard deviations used are listed in Table 1.

The initial data configuration examined was formed by dividing the data into four intervals, with seven points in the first interval and eight points in the remaining three intervals. The results of inverting the data with this configuration of the test data are presented in Table 2 and Figs. 2 and 3. Figure 2 displays the input radiance data as points and the results of the curve fit as a continuous line. Error intervals equal in magnitude to the radiance standard deviations are shown for each data point. Figure 3 displays the profile of the calculated emission coefficients along with the calculated error interval for each emission coefficient value.

Another configuration of the same set of test data was inverted after distributing the 31 data points into six intervals with five points in the first five intervals and six points in the sixth interval. The results of this case are presented in Table 3 and Figs. 4 and 5. Figure 4 shows the radiance and Fig. 5 the emission coefficient profile for this configuration of the data.

A comparison of Tables 1 and 2 reveals that the percentage errors between the curve fit data and the input data are generally smaller in magnitude for the six-interval configuration than for the four-interval configuration. Of the 31 percentage errors, 24 are smaller for the six-interval configuration. This suggests that the curve fit of the radiance data was better for the case of six intervals. Furthermore, of the 31 standard deviation intervals displayed in each of Figs. 3 and 5, 22 are smaller for the six-interval case. This fact is illustrated more clearly by comparing the calculated standard deviations of the emission coefficient values for the two cases which are listed in the last columns of Tables 2 and 3.

Table 1. Checkout Data with e^{-x^2} Values Used as Input Data

Displacement	Radiance (Data)	Standard Deviation
0.0	1.0000 E-00	1.0000 E-01
0.1	9.9005 E-01	9.9005 E-02
0.2	9.6079 E-01	9.6079 E-02
0.3	9.1393 E-01	9.1393 E-02
0.4	8.5214 E-01	8.5214 E-02
0.5	7.7880 E-01	7.7880 E-02
0.6	6.9768 E-01	6.9768 E-02
0.7	6.1263 E-01	6.1263 E-02
0.8	5.2729 E-01	5.2729 E-02
0.9	4.4860 E-01	4.4860 E-02
1.0	3.6788 E-01	3.6788 E-02
1.1	2.9820 E-01	2.9820 E-02
1.2	2.3693 E-01	2.3693 E-02
1.3	1.8452 E-01	1.8452 E-02
1.4	1.4086 E-01	1.4086 E-02
1.5	1.0540 E-01	1.0540 E-02
1.6	7.7305 E-02	7.7305 E-03
1.7	5.5576 E-02	5.5576 E-03
1.8	3.9164 E-02	3.9164 E-03
1.9	2.7052 E-02	2.7052 E-03
2.0	1.8316 E-02	1.8316 E-03
2.1	1.2155 E-02	1.2155 E-03
2.2	7.9071 E-03	7.9071 E-04
2.3	5.0418 E-03	5.0418 E-04
2.4	3.1511 E-03	3.1511 E-04
2.5	1.9305 E-03	1.9305 E-04
2.6	1.1592 E-03	1.1592 E-04
2.7	6.8233 E-04	6.8233 E-05
2.8	3.9367 E-04	3.9367 E-05
2.9	2.2263 E-04	2.2263 E-05
3.0	1.2341 E-04	1.2341 E-05

Table 2. Inversion Results Using Four Intervals of e^{-x^2} Values as Input Data

Displacement	Radiance (Calculated)	Percent Error between Radiance Data and Calculated Radiance	Emission Coefficient	Standard Deviation (Emission Coefficient)
0.0	1.000687 E-00	6.873565 E-02	5.675450 E-01	1.277898 E-01
1.000000 E-01	9.904464 E-01	4.003959 E-02	6.613535 E-01	1.183743 E-01
2.000000 E-01	9.604699 E-01	-3.331861 E-02	5.432904 E-01	9.255000 E-02
3.000000 E-01	9.128938 E-01	-1.133770 E-01	5.148417 E-01	5.781569 E-02
4.000000 E-01	8.509355 E-01	-1.413484 E-01	4.783169 E-01	3.000306 E-02
5.000000 E-01	7.783793 E-01	-5.401784 E-02	4.365562 E-01	3.155638 E-02
6.000000 E-01	6.988567 E-01	1.686561 E-01	3.923847 E-01	3.612286 E-02
7.000000 E-01	6.149943 E-01	3.859220 E-01	3.470961 E-01	2.810619 E-02
8.000000 E-01	5.293764 E-01	3.956902 E-01	3.001493 E-01	1.929250 E-02
9.000000 E-01	4.455566 E-01	-6.784139 E-01	2.531834 E-01	1.210634 E-02
1.000000 E-01	3.668708 E-00	-2.743241 E-01	2.081161 E-01	8.488109 E-03
1.100000 E-00	2.960949 E-00	-7.059348 E-01	1.668016 E-01	9.052147 E-03
1.200000 E-00	2.351273 E-00	-7.608546 E-01	1.308842 E-01	1.032227 E-03
1.300000 E-00	1.846380 E-00	6.396129 E-02	1.015773 E-01	9.970046 E-03
1.400000 E-00	1.436843 E-01	2.005025 E-00	7.925598 E-02	7.552663 E-03
1.500000 E-00	1.094268 E-01	3.820451 E-00	6.210448 E-02	4.133476 E-03
1.600000 E-00	8.000591 E-02	3.493838 E-00	4.684850 E-02	2.054649 E-03
1.700000 E-00	5.579709 E-02	3.978129 E-01	3.352474 E-02	2.698357 D-03
1.800000 E-00	3.703233 E-02	-5.442925 E-00	2.244468 E-02	3.813321 E-03
1.900000 E-00	2.362409 E-02	-1.267157 E-01	1.384293 E-02	4.234182 E-03
2.000000 E-00	1.510630 E-02	-1.752401 E-01	7.840132 E-03	3.858621 E-03
2.100000 E-00	1.057220 E-02	-1.057220 E-01	4.382294 E-03	2.596808 E-03
2.200000 E-00	8.606659 E-03	8.872517 E-00	3.113924 E-03	8.552391 E-04
2.300000 E-00	7.346027 E-03	4.570248 E-01	2.879137 E-03	9.776775 E-04
2.400000 E-00	5.995310 E-03	9.026087 E-01	2.571927 E-03	1.831465 E-03
2.500000 E-00	4.613937 E-03	1.390022 E-02	2.192267 E-03	1.821228 E-03
2.600000 E-00	3.264921 E-03	1.816530 E-02	1.750231 E-03	1.024300 E-03
2.700000 E-00	2.026027 E-03	1.969277 E-02	1.263463 E-03	3.872080 E-04
2.800000 E-00	9.912250 E-04	1.517908 E-02	7.616202 E-04	2.026932 E-03
2.900000 E-00	2.722152 E-04	2.227249 E-01	2.972067 E-04	3.192960 E-03
3.000000 E-00	0.00	-1.000000 E-02	0.0	0.0

Number of points: 31

Number of intervals: 4

Number of points per interval: 7 8 8 8

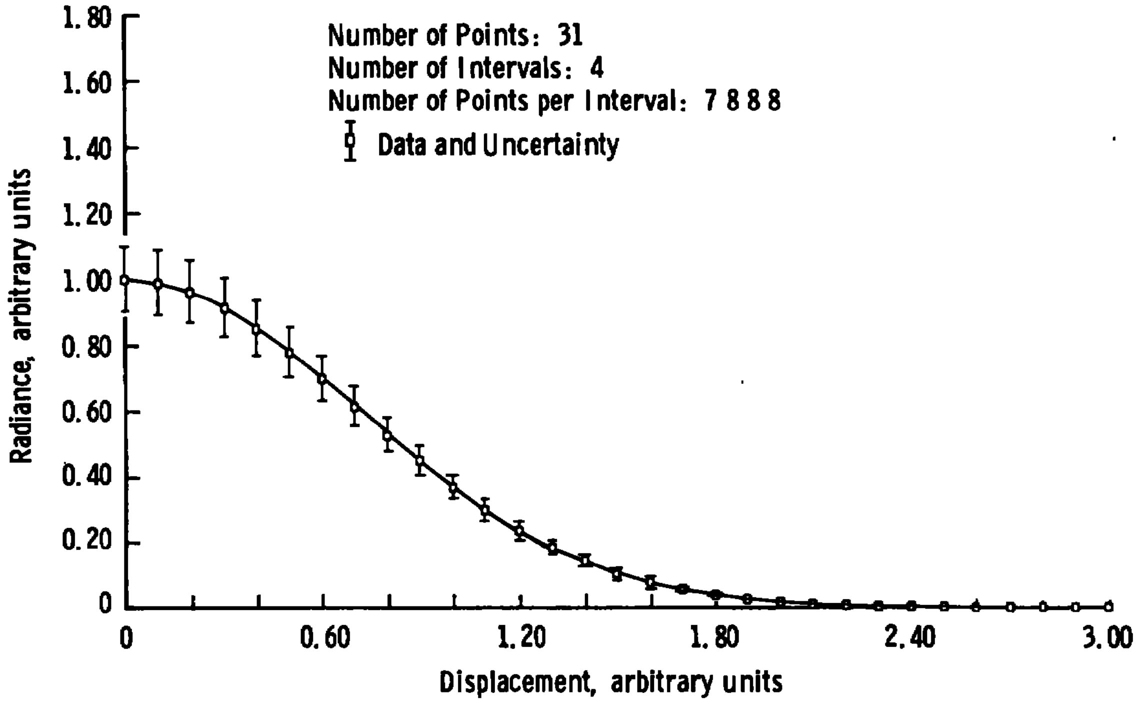

Figure 2. Intensity for four-interval e^{-x^2} test data.

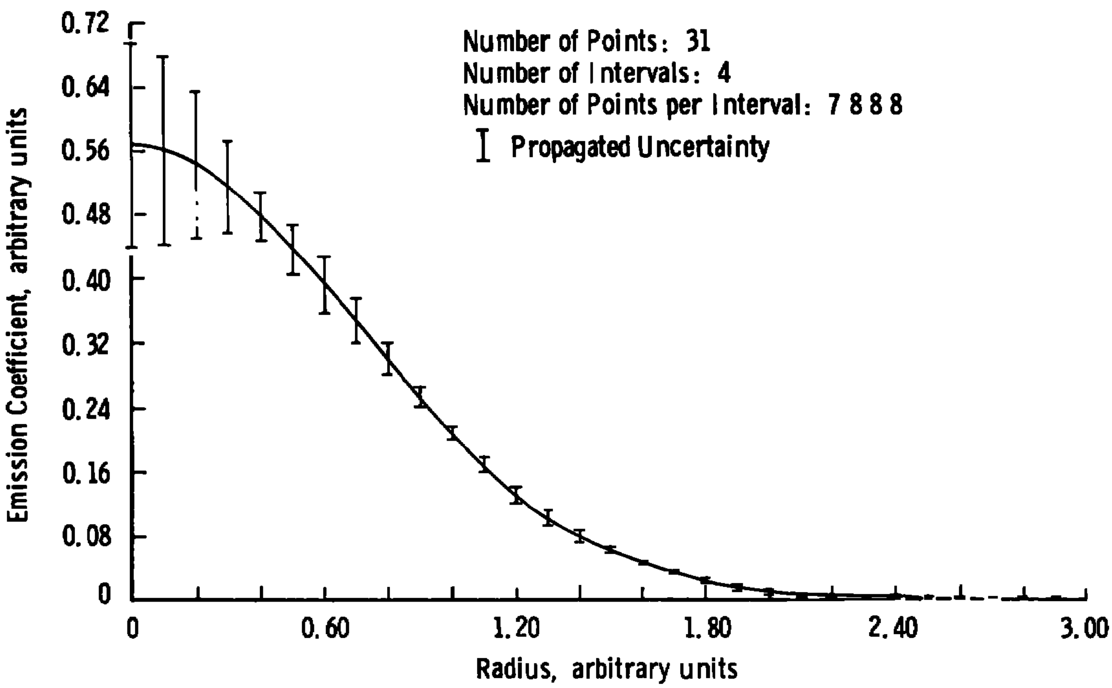

Figure 3. Emission coefficient profile for four-interval e^{-x^2} test data.

Table 3. Inversion Result with Six-Interval e^{-x^2} Values Used as Input Data

Displacement	Radiance (Calculated)	Percent Error between Radiance Data and Calculated Radiance	Emission Coefficient	Standard Deviation (Emission Coefficient)
0.0	1.000057 E-00	5.727384 E-03	5.644821 E-01	1.887621 E-01
1.000000 E-01	9.900608 E-01	1.095800 E-03	5.587444 E-01	1.589008 E-01
2.000000 E-01	9.607041 E-01	-8.935534 E-03	5.419311 E-01	8.748871 E-02
3.000000 E-01	9.138023 E-01	-1.397761 E-02	5.151949 E-01	4.705685 E-02
4.000000 E-01	8.521048 E-01	-4.125894 E-03	4.802804 E-01	6.393484 E-02
5.000000 E-01	7.788967 E-01	1.241464 E-02	4.390734 E-01	4.259269 E-02
6.000000 E-01	6.978503 E-01	2.441074 E-02	3.932836 E-01	2.312669 E-02
7.000000 E-01	6.129601 E-01	5.388051 E-02	3.452108 E-01	2.111331 E-02
8.000000 E-01	5.280066 E-01	1.359116 E-01	2.972415 E-01	2.471709 E-02
9.000000 E-01	4.460712 E-01	-5.636391 E-01	2.514763 E-01	2.046078 E-02
1.000000 E-00	3.690464 E-01	3.170582 E-01	2.088244 E-01	1.096935 E-02
1.100000 E-00	2.985861 E-01	1.294751 E-01	1.690972 E-01	7.882454 E-03
1.200000 E-00	2.366441 E-01	-1.206805 E-01	1.335991 E-01	9.789215 E-03
1.300000 E-00	1.842825 E-01	-1.287393 E-01	1.035031 E-01	9.862244 E-03
1.400000 E-00	1.413117 E-01	3.206904 E-01	7.938548 E-02	6.747721 E-03
1.500000 E-00	1.060471 E-01	6.139940 E-01	6.027553 E-02	2.732949 E-03
1.600000 E-00	7.724082 E-02	-8.302233 E-02	4.413603 E-02	2.944425 E-03
1.700000 E-00	5.489177 E-02	-1.231169 E-00	3.111812 E-02	2.669870 E-03
1.800000 E-00	3.854638 E-02	-1.577006 E-00	2.141045 E-02	3.670396 E-03
1.900000 E-00	2.711363 E-02	2.278071 E-01	1.488089 E-02	1.931244 E-03
2.000000 E-00	1.881522 E-02	2.725604 E-00	1.060456 E-02	6.662148 E-04
2.100000 E-00	1.246579 E-02	2.556850 E-00	7.207111 E-03	1.797132 E-03
2.200000 E-00	7.899407 E-03	-9.729131 E-02	4.604004 E-03	8.499760 E-04
2.300000 E-00	4.886028 E-03	-3.089606 E-00	2.794489 E-03	9.729830 E-04
2.400000 E-00	3.061859 E-03	-2.832055 E-00	1.712425 E-03	4.271174 E-04
2.500000 E-00	1.917642 E-03	-6.660687 E-01	1.137634 E-03	7.384108 E-04
2.600000 E-00	1.084741 E-03	-6.423344 E-0[illegible]	7.086608 E-04	1.008421 E-03
2.700000 E-00	5.185475 E-04	-2.400342 E-01	3.841813 E-04	3.792780 E-04
2.800000 E-00	1.843718 E-04	-5.316590 E-01	1.633415 E-04	3.629174 E-04
2.900000 E-00	3.315120 E-05	-8.510728 E-01	3.993210 E-05	1.356839 E-03
3.000000 E-00	0.0	-1.000000 E-02	0.0	0.0

Number of points. 31

Number of intervals: 6

Number of points per interval: 5 5 5 5 5 6

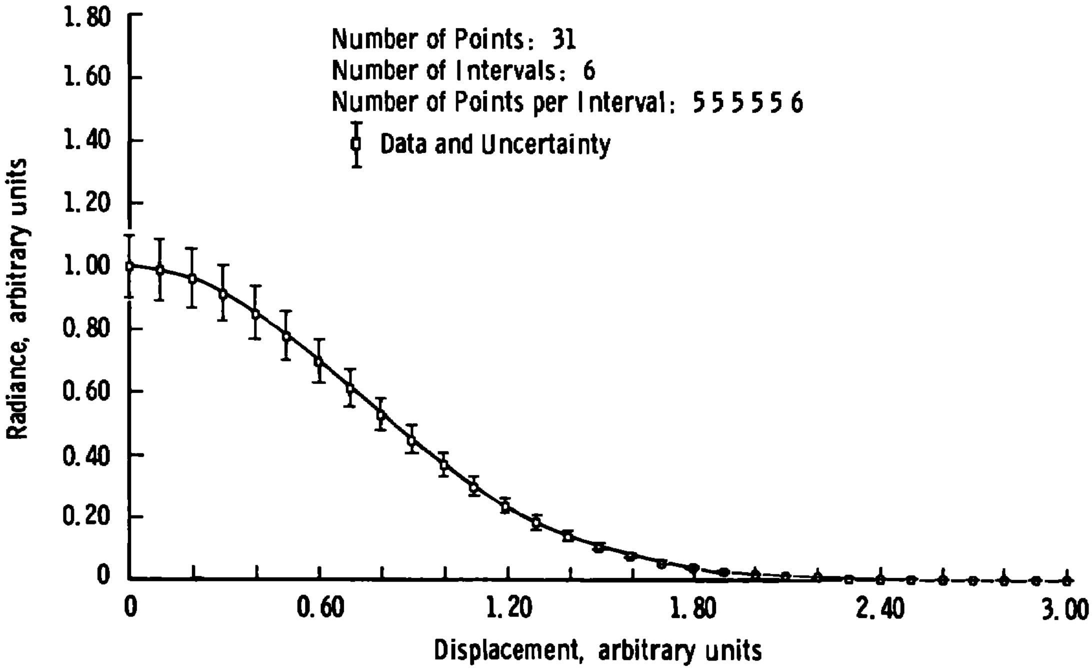

Figure 4. Intensity for six-interval e^{-x^2} test data.

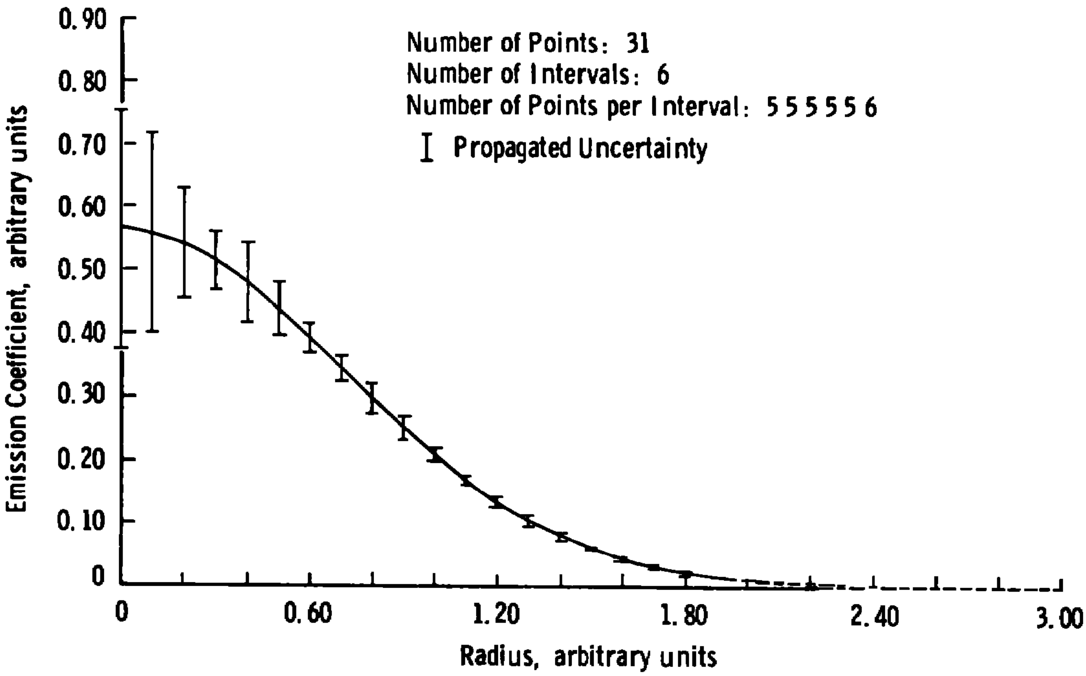

Figure 5. Emission coefficient profile for six-interval e^{-x^2} test data.

The results of this comparison point to the conclusion that the emission coefficient values obtained using the six-interval configuration are more accurate for the particular set of data tested than are the values obtained using the four-interval configuration. A comparison of the emission coefficient values obtained for the two cases with the analytically determined values obtained by evaluating Eq. (1) at the proper displacement values shows that this conclusion is indeed correct. Table 4 lists the analytical values of the emission coefficients for this particular problem and the percent error between the analytic emission coefficient values and the calculated values for the two cases. Of the 31 percentage errors listed for each configuration, 26 are smaller in magnitude for the six-interval case.

Table 5 and Figures 6 and 7 show the results of inverting the data with a configuration of six intervals with an uneven dispersion of the data points among the intervals--two points in the first, third, and fourth intervals, eight points in the second and fifth intervals, and nine points in the sixth interval. A comparison of these results with the results of the six-interval configuration with an even dispersion of the data (Table 3, Figs. 4 and 5) suggests that the configuration of data with the data points dispersed uniformly among the intervals yields more accurate emission coefficient values for these analytical data. This conclusion is supported by the data listed in Table 6, which shows the percentage error between the analytic emission coefficient values and the calculated values for the two cases. Examination of the standard deviations of the emission coefficients for the six-interval configuration with an even and uneven distribution of the data points among the intervals reveals that the even distribution provides noticeably smaller standard deviations near the centerline. Standard deviations are also generally smaller for larger displacement values for the even distribution, although the differences are not as great.

Table 4. Percentage Error Between Analytic Emission Coefficient Values and Calculated Values

Displacement	Analytic Emission Coefficient	Percent Error (Four Intervals)	Percent Error (Six Intervals)
0.0	5.634561 E-01	7.256821 E-01	1.820905 E-01
0.1	5.578497 E-01	6.280903 E-01	1.603831 E-01
0.2	5.413085 E-01	3.661313 E-01	1.150176 E-01
0.3	5.149086 E-01	-1.299259 E-02	5.560210 E-02
0.4	4.800495 E-01	-3.609211 E-01	4.809920 E-02
0.5	4.387322 E-01	-4.959745 E-01	7.776954 E-02
0.6	3.929919 E-01	-1.545070 E-01	7.425089 E-02
0.7	3.450153 E-01	-6.031037 E-01	5.666415 E-02
0.8	2.969277 E-01	1.084978 E-00	1.056823 E-01
0.9	2.504572 E-01	1.088489 E-00	4.068959 E-01
1.0	2.070764 E-01	5.020852 E-01	8.441329 E-01
1.1	1.678024 E-01	-5.964158 E-01	7.716219 E-01
1.2	1.332712 E-01	-1.791085 E-00	2.460397 E-01
1.3	1.037396 E-01	-2.084354 E-00	-2.279747 E-01
1.4	7.915297 E-02	1.301404 E-01	2.937477 E-01
1.5	5.918566 E-02	4.931634 E-00	1.841443 E-00
1.6	4.337465 E-02	8.008941 E-00	1.755357 E-00
1.7	3.114230 E-02	7.650174 E-00	-7.764359 E-02
1.8	2.191468 E-02	2.418470 E-00	-2.300878 E-00
1.9	1.510670 E-02	-8.365565 E-00	-1.494702 E-00
2.0	1.020432 E-02	-2.316850 E-01	3.922260 E-00
2.1	6.746728 E-03	-3.504564 E-01	6.823827 E-00
2.2	4.368733 E-03	-2.872249 E-01	5.385360 E-00
2.3	2.768276 E-03	4.004695 E-00	9.468709 E-01
2.4	1.714000 E-03	5.005408 E-01	-9.200700 E-02
2.5	1.103632 E-03	9.864114 E-01	3.080837 E-00
2.6	6.103361 E-04	1.867651 E-02	1.610981 E-01
2.7	3.483524 E-04	2.626968 E-02	1.028519 E-01
2.8	1.909872 E-04	2.987808 E-02	-1.447511 E-01
2.9	9.789680 E-05	2.035918 E-02	-5.917934 E-01
3.0	3.481320 E-05	-1.000000 E-02	-1.000000 E-02

Table 5. Inversion Results with Unevenly Dispersed e^{-x^2} Values Used as Input Data

Displacement	Radiance (Calculated)	Percent Error between Radiance Data and Calculated Radiance	Emission Coefficient	Standard Deviation (Emission Coefficient)
0.00	9.999282 E-01	-7.174047 E-03	5.609455 E-01	8.117026 E-01
1.000000 E-01	9.904682 E-01	4.224036 E-02	5.588918 E-01	1.611006 E-01
2.000000 E-01	9.610668 E-01	2.881206 E-02	5.431304 E-01	9.991750 E-02
3.000000 E-01	9.138335 E-01	-1.055980 E-02	5.161061 E-01	7.471066 E-02
4.000000 E-01	8.516753 E-01	-5.453786 E-02	4.804729 E-01	4.784840 E-02
5.000000 E-01	7.781568 E-01	-8.258385 E-02	4.382930 E-01	3.147328 E-02
6.000000 E-01	6.972846 E-01	-5.666698 E-02	3.920035 E-01	3.479315 E-02
7.000000 E-01	6.130743 E-01	7.251822 E-02	3.442192 E-01	3.938439 E-02
8.000000 E-01	5.290217 E-01	3.284228 E-01	2.974210 E-01	3.213198 E-02
9.000000 E-01	4.474787 E-01	-2.499491 E-01	2.533768 E-01	2.010979 E-02
1.000000 E-00	3.690851 E-01	3.275864 E-01	2.112019 E-01	2.918001 E-02
1.100000 E-00	2.960012 E-01	-7.373671 E-01	1.683103 E-01	2.209935 E-02
1.200000 E-00	2.339679 E-01	-1.250211 E-00	1.294129 E-01	8.216915 E-03
1.300000 E-00	1.850101 E-01	2.656128 E-01	1.011093 E-01	1.103693 E-02
1.400000 E-00	1.443999 E-01	2.513067 E-00	8.076948 E-02	7.790830 E-03
1.500000 E-00	1.088006 E-01	3.226334 E-00	6.254847 E-02	4.444626 E-03
1.600000 E-00	7.871845 E-02	1.828408 E-00	4.633172 E-02	2.182256 E-03
1.700000 E-00	5.457954 E-02	-1.792976 E-00	3.252025 E-02	1.997622 E-03
1.800000 E-00	3.646029 E-02	-6.903555 E-00	2.143476 E-02	2.740380 E-03
1.900000 E-00	2.401706 E-02	-1.121892 E-01	1.327453 E-02	2.979927 E-03
2.000000 E-00	1.641107 E-02	-1.040037 E-01	8.051607 E-03	2.499504 E-03
2.100000 E-00	1.222921 E-02	6.105477 E-01	5.459507 E-03	1.413835 E-03
2.200000 E-00	9.487126 E-03	1.998237 E-01	4.376879 E-03	3.050172 E-04
2.300000 E-00	7.102601 E-03	4.087430 E-01	3.451966 E-03	7.716312 E-04
2.400000 E-00	5.054098 E-03	6.070890 E-01	2.017529 E-03	1.100082 E-03
2.500000 E-00	3.383133 E-03	7.524648 E-01	1.883982 E-03	9.757774 E-04
2.600000 E-00	2.061617 E-03	7.784823 E-01	1.260428 E-03	4.451294 E-04
2.700000 E-00	1.090593 E-03	5.983366 E-01	7.543510 E-04	4.054118 E-04
2.800000 E-00	4.489278 E-04	1.403659 E-01	3.714054 E-04	1.337966 E-03
2.900000 E-00	1.019430 E-04	-5.420969 E-01	1.149017 E-04	1.959461 E-03
3.000000 E-00	0.0	-1.000000 E-02	0.0	0.0

Number of points: 31

Number of intervals: 6

Number of points per interval: 2 8 2 2 8 9

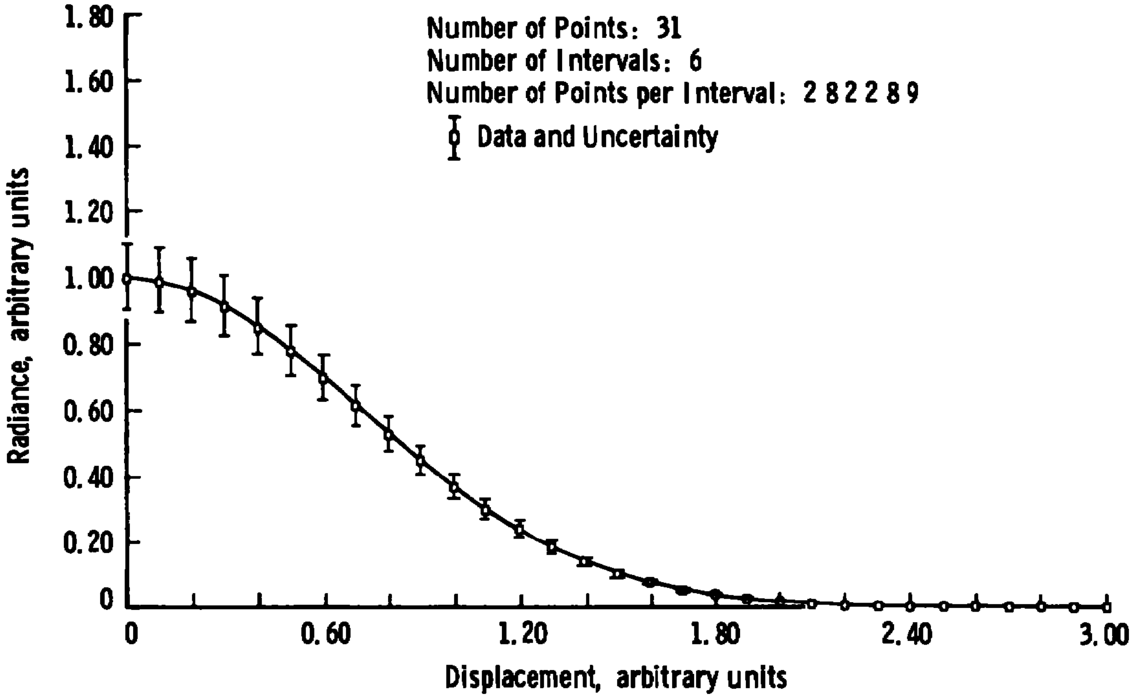

Figure 6. Intensity for unevenly dispersed e^{-x^2} test data.

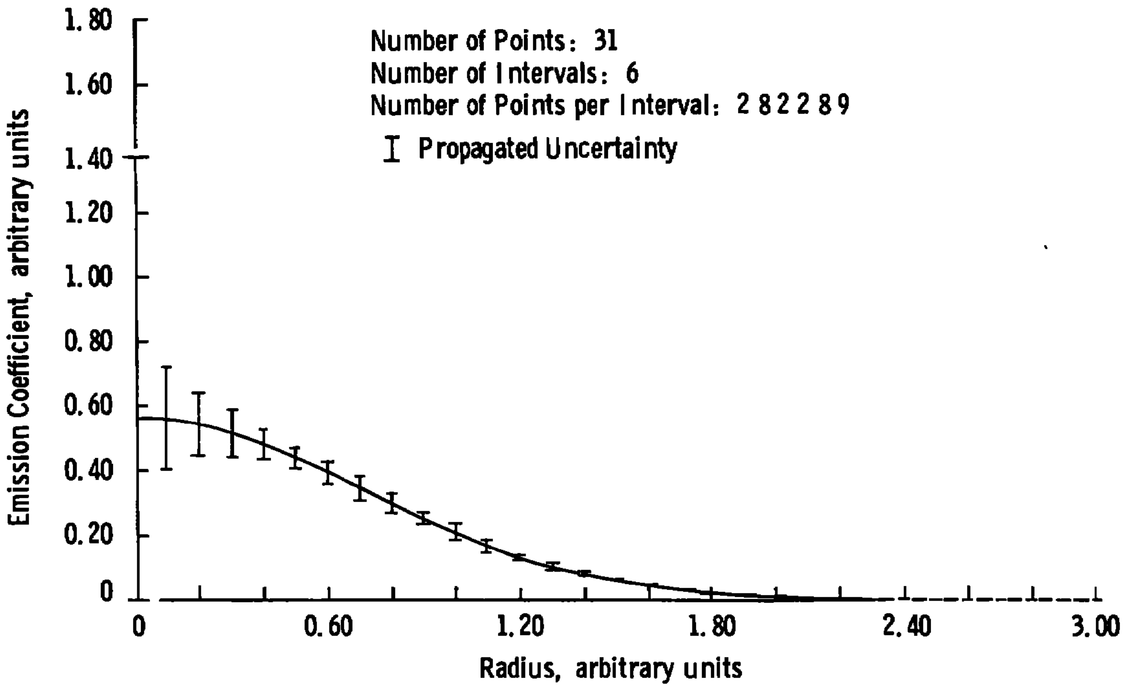

Figure 7. Emission coefficient profile for unevenly dispersed e^{-x^2} test data.

Table 6. Percentage Error Between Analytic Emission Coefficient Values and Calculated Values, Even and Uneven Data

Displacement	Percent Error (Even Dispersion)	Percent Error (Uneven Dispersion)
0.0	1.820905 E-01	-4.455715 E-01
0.1	1.603837 E-01	1.858055 E-01
0.2	1.150176 E-01	3.365733 E-01
0.3	5.560210 E-02	2.325655 E-01
0.4	4.809920 E-02	8.819924 E-02
0.5	7.776954 E-02	-1.001066 E-01
0.6	7.425089 E-02	-2.515065 E-01
0.7	5.666415 E-02	-2.307434 E-01
0.8	1.056823 E-01	-1.661347 E-01
0.9	4.068959 E-01	1.165708 E-00
1.0	8.441329 E-01	-1.992260 E-00
1.1	7.716219 E-01	3.026774 E-01
1.2	2.460397 E-01	-2.895074 E-00
1.3	-2.279747 E-01	-2.535483 E-00
1.4	2.937477 E-01	2.042261 E-00
1.5	1.841443 E-00	5.681799 E-00
1.6	1.755357 E-00	6.817507 E-00
1.7	-7.764359 E-02	4.424689 E-00
1.8	-2.300878 E-00	-2.189948 E-00
1.9	-1.494702 E-00	-1.212814 E-01
2.0	3.922260 E-00	-2.109521 E-01
2.1	6.823827 E-00	-1.907919 E-01
2.2	5.385360 E-00	1.864614 E-01
2.3	9.468709 E-01	2.469732 E-01
2.4	-9.200700 E-02	5.271464 E-01
2.5	3.080837 E-00	7.070746 E-01
2.6	1.610981 E-01	1.065138 E-02
2.7	1.028519 E-01	1.165482 E-02
2.8	-1.447511 E-01	9.446612 E-01
2.9	-5.917934 E-01	1.839171 E-01
3.0	-1.000000 E-02	-1.000000 E-02

As a further examination of the technique, the 31 ordinates generated from the function of Eq. (33) were randomly scattered by a fractional amount bounded by ±10 percent of the radiance values. The scattered data, listed in Table 7, were tested for all the same parameter configurations used for the unscattered data. Table 8 lists the results of inverting the scattered data in the six-interval configuration, and Figs. 8 and 9 display the results graphically. A comparison of this data with that shown in

Table 7. Checkout Data with Scattered e^{-x^2} Values Used as Input Data

Displacement	Intensity (Data)	Standard Deviation
0.0	9.90000 E-01	9.90000 E-02
0.1	9.30650 E-01	9.30650 E-02
0.2	1.01844 E-00	1.01844 E-01
0.3	9.23070 E-01	9.23070 E-02
0.4	8.60660 E-01	8.60660 E-02
0.5	7.24284 E-01	7.24284 E-02
0.6	6.97680 E-01	6.97680 E-02
0.7	5.88120 E-01	5.88120 E-02
0.8	5.43110 E-01	5.43110 E-02
0.9	4.08230 E-01	4.08230 E-02
1.0	3.45810 E-01	3.45810 E-02
1.1	2.98200 E-01	2.98200 E-02
1.2	2.55880 E-01	2.55880 E-02
1.3	1.95590 E-01	1.95590 E-02
1.4	1.31000 E-01	1.31000 E-02
1.5	1.10670 E-01	1.10670 E-02
1.6	7.49860 E-02	7.49860 E-03
1.7	5.89110 E-02	5.89110 E-03
1.8	3.56390 E-02	3.56390 E-03
1.9	2.62400 E-02	2.62400 E-03
2.0	1.74000 E-02	1.74000 E-03
2.1	1.13040 E-02	1.13040 E-03
2.2	8.06520 E-03	8.06520 E-04
2.3	4.89050 E-03	4.89050 E-04
2.4	2.86750 E-03	2.86750 E-04
2.5	1.81470 E-03	1.81470 E-04
2.6	1.10120 E-03	1.10120 E-04
2.7	7.30090 E-04	7.30090 E-05
2.8	3.93670 E-04	3.93670 E-05
2.9	2.20400 E-04	2.20400 E-05
3.0	1.25880 E-04	1.25880 E-05

Table 8. Inversion Results with Randomly Scattered e^{-x^2} Values Used as Input

Displacement	Radiance (Calculated)	Percent Error between Radiance Data and Calculated Radiance	Emission Coefficient	Standard Deviation (Emission Coefficient)
0.0	9.789692 E-01	-1.114224 E-00	4.841228 E-01	2.265459 E-01
1.000000 E-01	9.761542 E-01	4.889508 E-00	4.952562 E-01	1.904222 E-01
2.000000 E-01	9.627751 E-01	-5.465700 E-00	5.203799 E-01	1.042947 E-01
3.000000 E-01	9.271917 E-01	4.465221 E-01	5.371826 E-01	5.780840 E-02
4.000000 E-01	8.605448 E-01	-1.338141 E-02	5.183171 E-01	7.889482 E-02
5.000000 E-01	7.715935 E-01	5.531897 E-00	4.591535 E-01	5.243160 E-02
6.000000 E-01	6.781933 E-01	-2.793076 E-00	3.938677 E-01	2.803216 E-02
7.000000 E-01	5.875534 E-01	-9.633717 E-02	3.313019 E-01	2.527556 E-02
8.000000 E-01	5.044674 E-01	-7.115067 E-00	2.764865 E-01	2.979423 E-02
9.000000 E-01	4.305600 E-01	5.469958 E-00	2.327488 E-01	2.474729 E-02
1.000000 E-00	3.628808 E-01	4.936454 E-00	1.578665 E-01	1.315299 E-02
1.100000 E-00	2.990449 E-01	2.833262 E-01	1.648065 E-01	9.276026 E-03
1.200000 E-00	2.406352 E-01	-5.957774 E-00	1.341215 E-01	1.165493 E-02
1.300000 E-00	1.889666 E-01	-3.386384 E-00	1.064485 E-01	1.183559 E-02
1.400000 E-00	1.445752 E-01	1.043906 E-01	8.252229 E-02	8.159877 E-03
1.500000 E-00	1.075517 E-01	-2.808620 E-00	6.227366 E-02	3.398379 E-03
1.600000 E-00	7.739311 E-02	3.210078 E-00	4.511455 E-02	3.554770 E-03
1.700000 E-00	5.420396 E-02	-7.900078 E-00	3.134969 E-02	4.824774 E-03
1.800000 E-00	3.748272 E-02	5.173332 E-00	2.117924 E-02	4.383155 E-03
1.900000 E-00	2.502468 E-02	-8.205769 E-01	1.445394 E-02	2.311607 E-03
2.000000 E-00	1.787807 E-02	2.747514 E-00	1.018137 E-02	7.933870 E-04
2.100000 E-00	1.173791 E-02	3.838573 E-00	6.799439 E-03	2.135224 E-03
2.200000 E-00	7.410726 E-03	-8.114787 E-00	4.269860 E-03	2.525229 E-03
2.300000 E-00	4.635326 E-03	-5.217757 E-00	2.564192 E-03	1.885208 E-03
2.400000 E-00	3.007434 E-03	4.870997 E-00	1.600111 E-03	5.088279 E-04
2.500000 E-00	1.974182 E-03	8.843470 E-00	1.113705 E-03	8.781242 E-04
2.600000 E-00	1.187082 E-03	7.738858 E-00	7.347368 E-04	1.281554 E-03
2.700000 E-00	6.170468 E-04	-1.548346 E-01	4.318422 E-04	7.290620 E-04
2.800000 E-00	2.482553 E-04	-3.693822 E-01	2.075203 E-04	4.318742 E-04
2.900000 E-00	5.465260 E-05	-7.520300 E-01	6.258467 E-05	1.614008 E-03
3.000000 E-00	0.0	-1.000000 E-02	0.0	0.0

Number of points: 31
Number of intervals: 6
Number of points per interval: 5 5 5 5 5 6

Table 3 and in Figs. 4 and 5 for the case of the unscattered data in the same six-interval configuration indicates that, as would be expected, the results of the radiance curve fit and the emission coefficient calculations were more accurate for the case of the unscattered data. This conclusion is supported further by the data listed in Table 9, which shows the percentage error between the analytic emission coefficient values and the calculated values for the two cases. Of the 31 values listed for each case, 27 are smaller in magnitude for the case of the unscattered data.

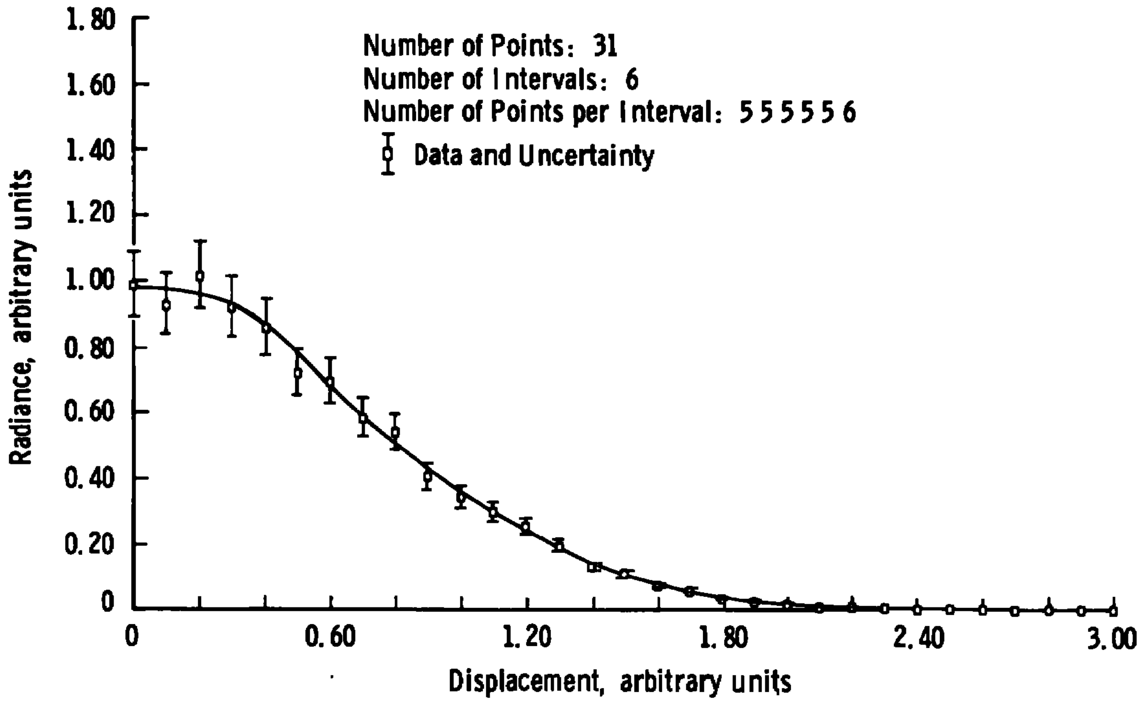

Figure 8. Intensity for scattered e^{-x^2} test data.

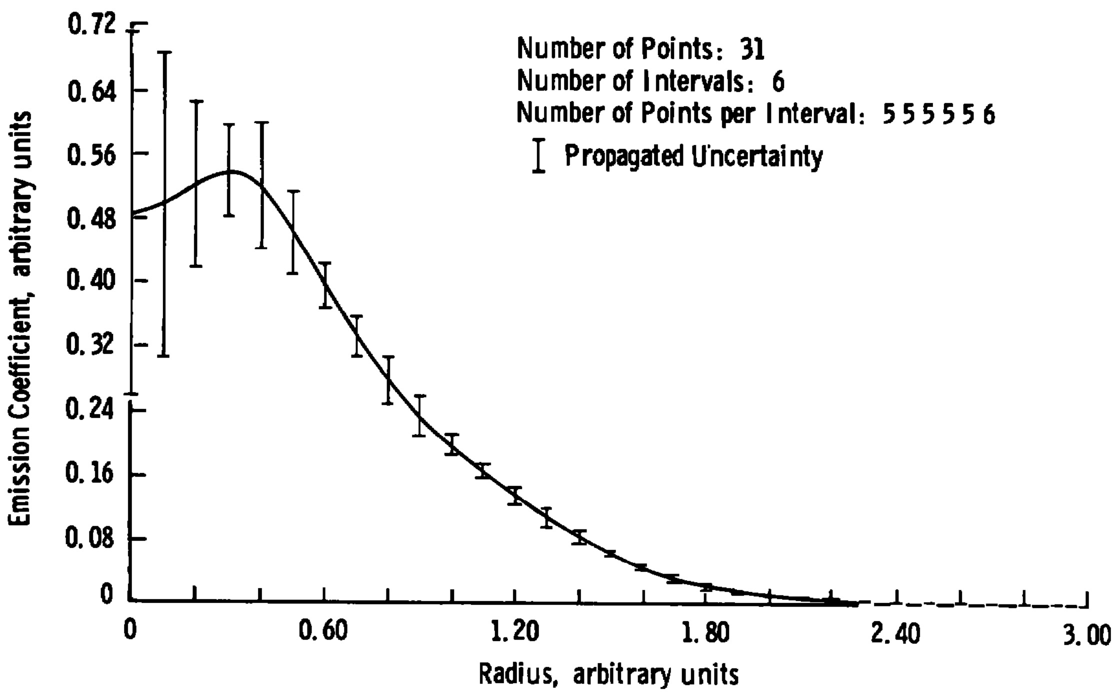

Figure 9. Emission coefficient profile for scattered e^{-x^2} test data.

Table 9. Percentage Error Between Analytic Emission Coefficient Values and Calculated Values, Scattered and Unscattered Data

Displacement	Percent Error Scattered Data	Percent Error Unscattered Data
0.0	-1.407971 E-01	1.820905 E-01
0.1	-1.122049 E-01	1.603837 E-01
0.2	-3.866298 E-00	1.150176 E-01
0.3	-4.325816 E-00	5.560210 E-02
0.4	-7.971595 E-00	4.809920 E-02
0.5	4.658263 E-00	7.776954 E-02
0.6	-2.228545 E-01	7.425089 E-02
0.7	-3.974722 E-00	5.666415 E-02
0.8	-6.884235 E-00	1.056823 E-01
0.9	-7.074422 E-00	4.068959 E-01
1.0	-4.447586 E-00	8.441329 E-01
1.1	-1.729951 E-00	7.716219 E-01
1.2	6.380223 E-01	2.460397 E-01
1.3	2.611250 E-00	-2.279747 E-01
1.4	4.256720 E-00	2.937477 E-01
1.5	5.217480 E-00	1.841442 E-00
1.6	4.011329 E-00	1.755357 E-00
1.7	6.659431 E-00	-7.764359 E-02
1.8	-3.355924 E-00	-2.300878 E-00
1.9	8.567907 E-00	-1.494702 E-00
2.0	-4.209002 E-01	3.922260 E-00
2.1	7.821717 E-00	6.823872 E-00
2.2	-2.263196 E-00	5.385360 E-00
2.3	-7.372242 E-00	9.468709 E-01
2.4	-6.644632 E-00	-9.200700 E-02
2.5	9.127228 E-01	3.080837 E-00
2.6	2.038233 E-01	1.610981 E-01
2.7	2.396705 E-01	1.028519 E-01
2.8	8.656653 E-00	-1.447511 E-01
2.9	-3.607077 E-01	-5.917934 E-01
3.0	-1.000000 E-02	-1.000000 E-02

3.3 TEST DATA WITH DISCONTINUITY

It is the nature of least-squares curve fitting techniques to reduce fluctuations in the data. Indeed, the techniques are designed to possess this characteristic, making them a highly useful tool for reducing the effects of experimental scatter in the input data. However, when a phenomenon such as a sudden change in slope is a vital aspect of the raw data and therefore an important feature of the emission coefficient values to be calculated, care must be taken in manipulating the parameters to achieve a data set configuration to attain optimum accuracy in the resultant emission coefficient values. Since the basis of the inversion technique is a polynomial curve fit, it is reasonable to expect that a good radiance curve fit will be difficult to achieve when the data possess an abrupt change in slope. To test this hypothesis, a set of data was constructed by using the following form for the emission coefficient function:

$$\epsilon(r) = \begin{cases} 0.5 , & 0 \leq r \leq 15 \\ \dfrac{25}{r^2} , & 15 \leq r \leq 20 \end{cases} \tag{35}$$

By evaluating Eq. (1) for the above functional form of $\varepsilon(r)$, one finds the associated analytical radiance function to be

$$I(x) = \begin{cases} (400 - x^2)^{1/2} , & 0 \leq x \leq 15 \\ \dfrac{50}{x} \sec^{-1} \left| \dfrac{20}{x} \right| , & 15 < x \leq 20 \end{cases} \tag{36}$$

Twenty-one data points were generated for input data using Eq. (36). These input data are listed in Table 10 along with the associated values of the emission coefficient found by evaluating Eq. (35) at the appropriate displacement values. Note that the sharp drop in magnitude of the radiance between the displacement values of 15 and 16 is reflected in the emission coefficient values at the same points. The initial radiance data are shown graphically in Fig. 10, whereas Fig. 11 displays the emission

coefficient profile. The dotted line in Fig. 11 represents the analytic results, Eq. (35). The smoothing process created a curve which gives results quite different from the analytical results, as would be expected from Fig. 10.

Table 10. Input Data Defining a Curve with a Sudden Change in Slope

Displacement	Radiance (Data)	Analytic Emission Coefficient
0	20.00000	0.50000000
1	19.97498	0.50000000
2	19.89975	0.50000000
3	19.77372	0.50000000
4	19.59592	0.50000000
5	19.36492	0.50000000
6	19.07878	0.50000000
7	18.73499	0.50000000
8	18.33030	0.50000000
9	17.86057	0.50000000
10	17.32051	0.50000000
11	16.70329	0.50000000
12	16.00000	0.50000000
13	15.19868	0.50000000
14	14.28286	0.50000000
15	13.22876	0.50000000
16	2.01094	0.09765626
17	1.63180	0.08650519
18	1.25285	0.07716049
19	0.83569	0.06925208
20	0.00000	0.06250000

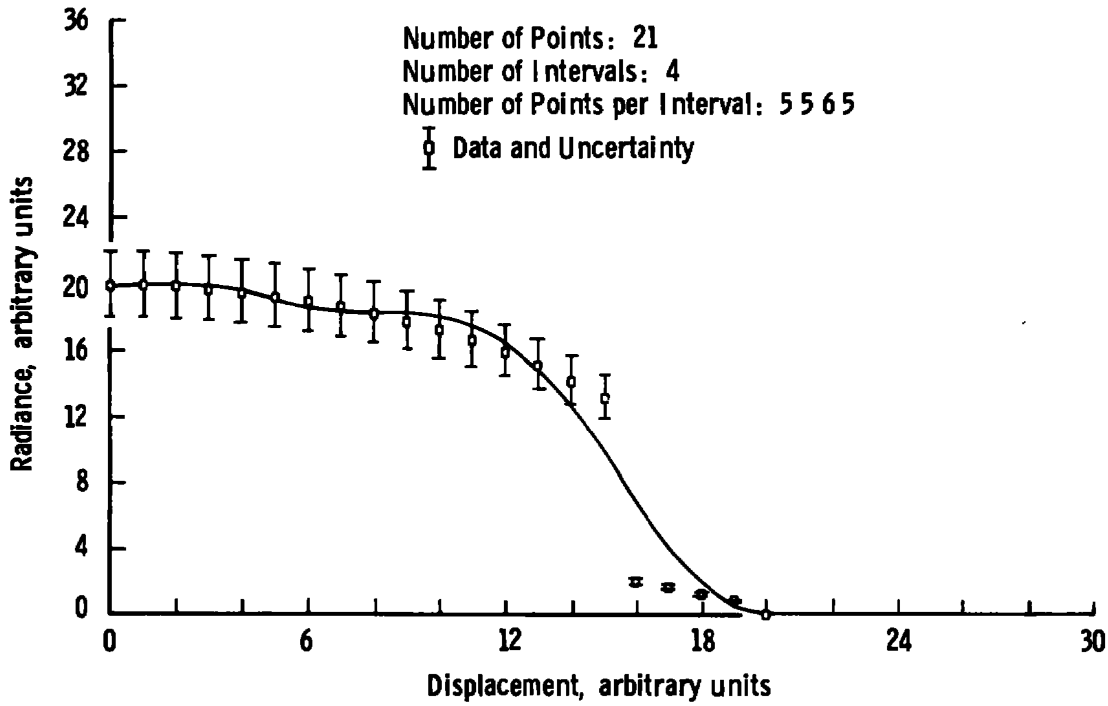

Figure 10. Intensity for twenty-one point sudden change in slope data.

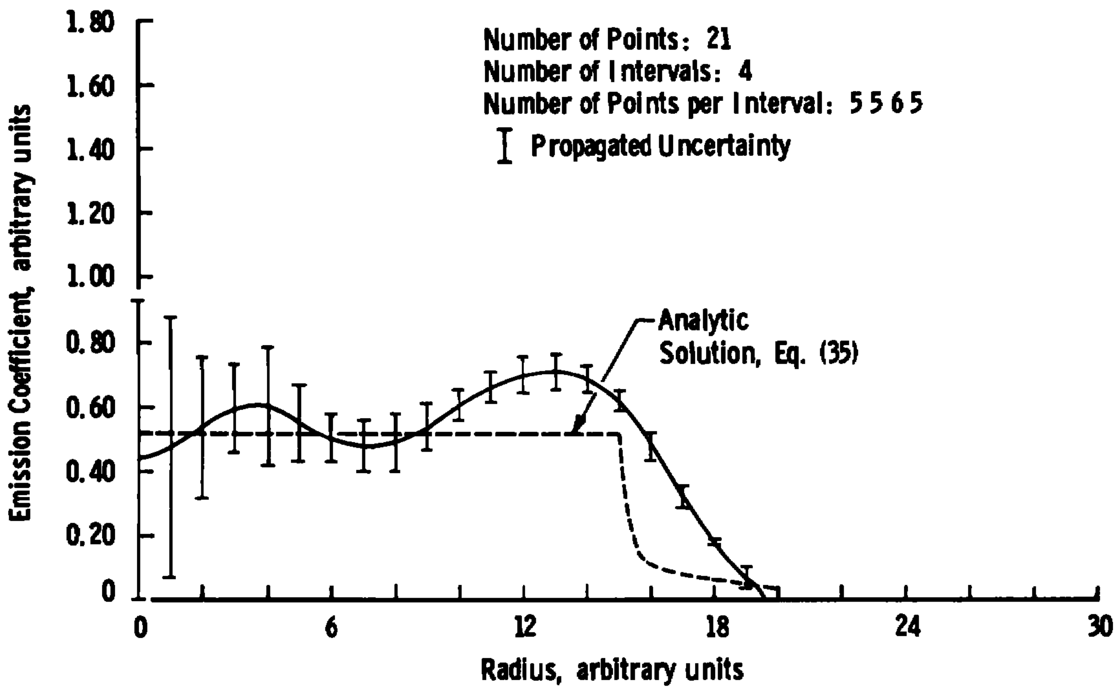

Figure 11. Emission coefficient profile for twenty-six point sudden change in slope test data.

In an attempt to better reproduce the steeply sloping portion of the data between the displacement values of 15 and 20, five more points within the interval $15 \leq x \leq 20$ were generated from Eq. (36). This set of data is listed in Table 11 along with the associated analytic values of the emission coefficients. The initial radiance data are shown graphically in Fig. 12, and Fig. 13 displays the emission coefficient profile obtained by using the 26 data points listed in Table 11 as well as the analytic results, Eq. (35). A comparison of Figs. 11 and 13 shows that the sudden change in slope in the emission coefficient curve is better reproduced using the second set of data. Nevertheless, the overall results of the inversion are quite unacceptable and seem no better than the results illustrated in Fig. 11.

Table 11. Twenty-Six Input Data Points Defining a Curve with a Sudden Change in Slope

Displacement	Radiance (Data)	Analytic Emission Coefficient
0.0	20.00000	0.50000000
1.0	19.97498	0.50000000
2.0	19.89975	0.50000000
3.0	19.77372	0.50000000
4.0	19.59592	0.50000000
5.0	19.36492	0.50000000
6.0	19.07878	0.50000000
7.0	18.73499	0.50000000
8.0	18.33030	0.50000000
9.0	17.86057	0.50000000
10.0	17.32051	0.50000000
11.0	16.70329	0.50000000
12.0	16.00000	0.50000000
13.0	15.19868	0.50000000
14.0	14.28286	0.50000000
15.0	13.22876	0.50000000
15.5	2.20671	0.10405830
16.0	2.01094	0.09765626
16.5	1.81998	0.09182736
17.0	1.63180	0.08650519
17.5	1.44389	0.08163265
18.0	1.25285	0.07716049
18.5	1.05341	0.07304602
19.0	0.83569	0.06925208
19.5	0.57455	0.06574622
20.0	0.00000	0.06250000

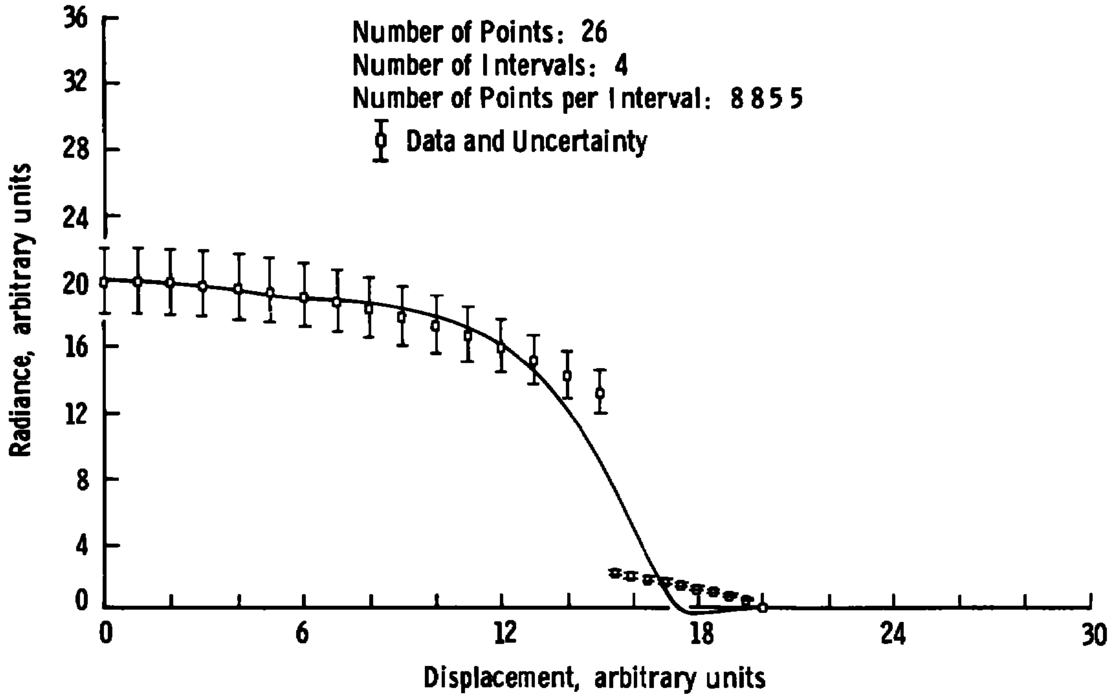

Figure 12. Intensity for twenty-one point sudden change in slope data.

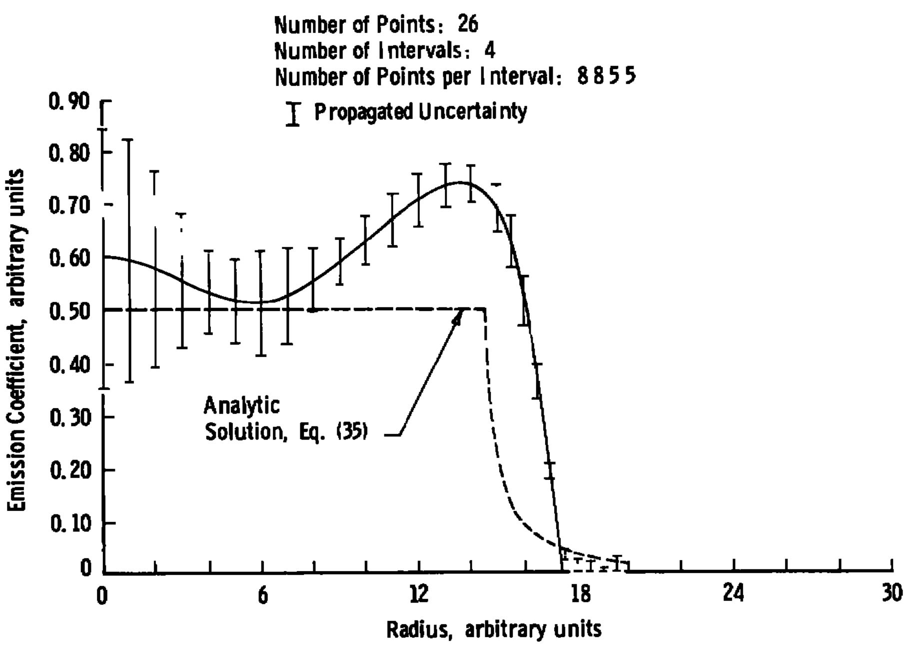

Figure 13. Emission coefficient profile for twenty-six point sudden change in slope test data.

It should be noted from examination of Figs. 10 and 12 that the portion of the radiance curve possessing the steep slope is not well reproduced in either case. In fact, portions of the fitted curve in the steep slope region lie entirely outside the standard deviation error bounds on the radiance data.

It is clear from these examples that it is not always possible to adequately fit the data by the use of the least-squares spline fit technique described herein. It is reasonable to expect that when the data are not well fitted the resultant emission coefficients will not be accurate. This conclusion is supported by examination of the data in Tables 10 and 11 and in Figs. 11 and 13, which deal with the emission coefficients for the radiance data of Figs. 10 and 12. Consequently, in such cases, a different approach to the inversion or a different set of fitting functions, something which has the potential of modeling the discontinuity, should be used.

3.4 APPLICATION TO EXPERIMENTAL DATA

As an illustration of the application of the inversion technique to typical experimental data, inversion results for the radiance profile of a selected spectral line from an argon arcjet are presented. The details of the experiment and implications of the results are included in Ref. 16. The radiance profile is shown in Fig. 14. Data from both sides of the centerline are included to indicate the symmetry of the plume and to provide additional data for the least-squares curve fit. The bars shown in Fig. 14 represent a typical two-standard-deviation bound for the data, and the curve represents the results of the least-squares spline fit. The results of the inversion and error propagation are shown in Fig. 15. The error bars represent the two-standard-deviation uncertainty at each of the radii. The largest standard deviation, that on the centerline, represents approximately a 10-percent uncertainty in the corresponding emission coefficient. The results represent the apparent best fit to the data, as determined by the propagated error, and represent physically acceptable results.

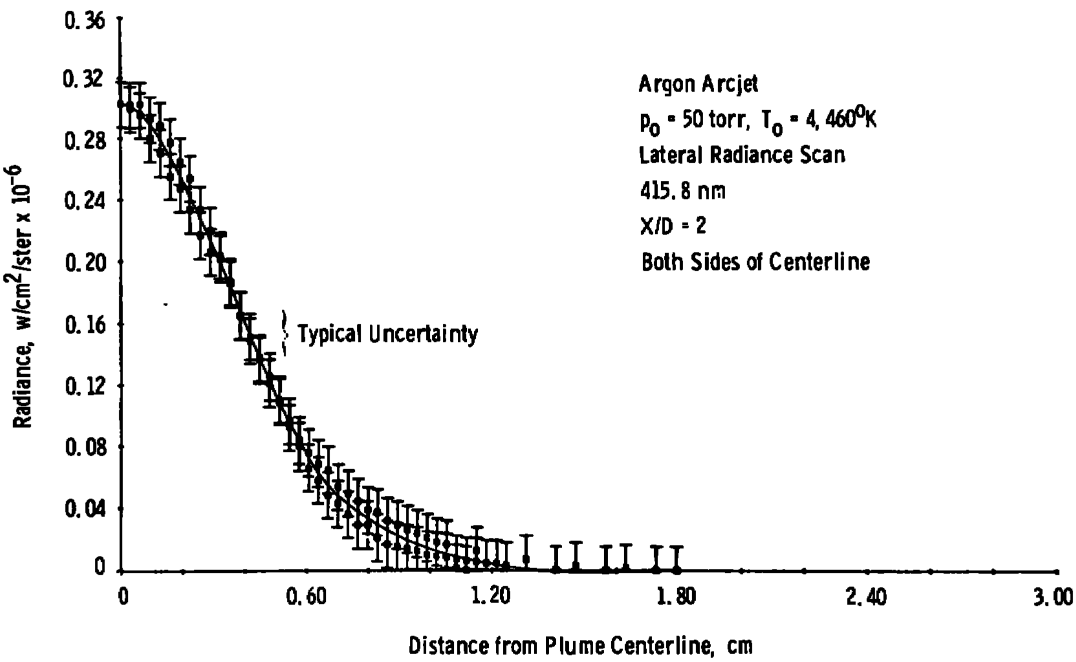

Figure 14. Typical lateral radiance scan from argon arcjet at X/D = 2, 415.8 nm.

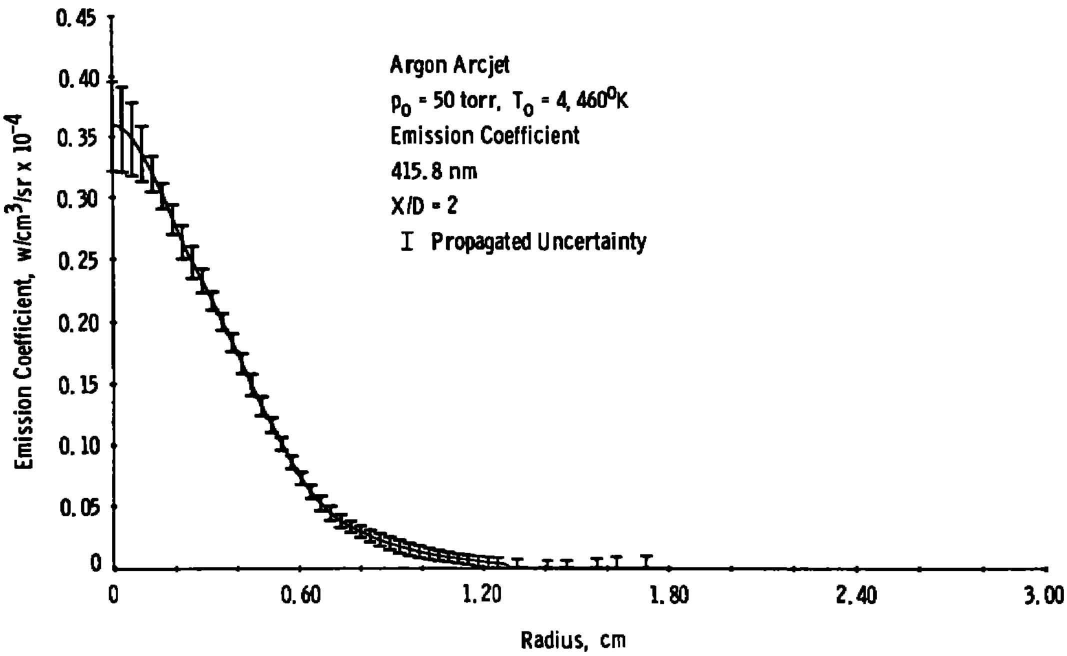

Figure 15. Typical emission coefficient profile from argon arcjet at X/D = 2, 415.8 nm.

4.0 SUMMARY

A method of performing Abel inversions and a method of determining the propagation of the associated experimental errors has been presented. The Abel inversion is applied to the problem of determining the radial distribution of emission coefficients from observations of the radiance from a cylindrically symmetric radiating source. The particular scheme for solving the Abel inversion problem is a least-squares polynomial spline fit technique. The spline fit technique involves the division of the raw data into several intervals and the least-squares curve fitting of the data points in each interval to a sixth-degree even polynomial. The ordinates, slopes, and second derivatives of the polynomials are required to be continuous at the interval boundaries. Thus the polynomials are constrained so that the total data profile is smooth and yet provides the best fit of all the data in the least-squares sense. The Abel inversion of the resultant polynomial model of the data is obtained analytically.

The associated error propagation analysis is developed by casting the numerical equations selected to perform the curve fit and integration into a form in which the problem can be viewed as a linear transformation from the raw data to the inverted results. In this manner, the variance-covariance matrix of the raw data can be directly transformed to the variance-covariance matrix of the emission coefficients. The result of the error propagation analysis provides an objective basis for the subjective determination of the series of polynomials providing the most nearly correct fit to the raw data and resultant emission coefficient. A computer program to perform the least-squares spline fit and associated error propagation analysis is described in Appendix A.

To determine an acceptable least-squares polynomial spline fit for a particular set of data, there are generally four parameters to be considered: (1) the total number of data points, (2) the number of

intervals into which the data are divided, (3) the number of points per interval, and (4) the displacement distribution of the data points. It has been shown that the parameter configurations of randomly scattered data which yield accurate emission coefficient values are generally the same parameter configurations which yield accurate emission coefficients for data without scatter. However, when the data curve possesses an abrupt change in slope, it is generally not possible, using a polynomial function, to arrange the data parameters into a configuration that yields an acceptable curve fit.

Although the development of the error propagation method has been applied specifically to a particular least-squares spline fit scheme, the method can be applied, with appropriate modifications, to any least-squares polynomial approximation or polynomial spline fit technique. Therefore, the error analysis technique can serve as a means of comparing the applicability of various schemes to the same problem.

It should be noted that each set of experimental data is unique, and the choice of data parameters must be based upon an analysis of the standard deviations generated for each data set. The error analysis process described provides an objective basis upon which a subjective judgement can be made concerning the acceptability of the results obtained by the application of the least-squares polynomial spline fit technique to a particular problem.

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APPENDIX A
COMPUTER PROGRAM DOCUMENTATION

A.1.0 GENERAL INFORMATION

The analytic and numerical approach described in the text of this report has been coded into a computer program to effect the Abel inversion of data from cylindrically symmetric sources and perform the associated propagation of experimental errors. The purpose of this appendix is to provide the description, documentation, and user manual for the computer program. The program described herein is in an "as developed" state and can be readily modified, as required, for more efficient operation for Abel inversion of data other than emission data.

A.1.1 DESCRIPTION OF PROBLEM

The physical problem is the determination of the radial distribution of the emission coefficients from measurements of the radiance from a cylindrically symmetric, optically thin radiating source. The problem, illustrated in Fig. A-1, is generally expressed mathematically as

$$y(x) = 2 \int_x^R \frac{\epsilon(r)\, r\, dr}{(r^2 - x^2)^{1/2}}$$

where y(x) is the measured radiance as a function of the displacement x, R is the overall radius of the source, and $\varepsilon(r)$ is the radially dependent emission coefficient, to be determined. The quantity y(x) is the usual experimental measurement. In the situation described, the emission coefficient can be expressed as

$$\epsilon(r) = -\frac{1}{\pi} \int_r^R \frac{(dy/dx)\, dx}{(x^2 - r^2)^{1/2}}$$

However, because y(x) is subject to experimental uncertainty, there is uncertainty in the derivative dy/dx. Furthermore, variations in dy/dx can have pronounced effects upon $\varepsilon(r)$ if the evaluation proceeds directly using raw data. Consequently, a smoothing process for the data coupled with a means of determining the effects of propagating the experimental uncertainty through the smoothing and subsequent inversion is required.

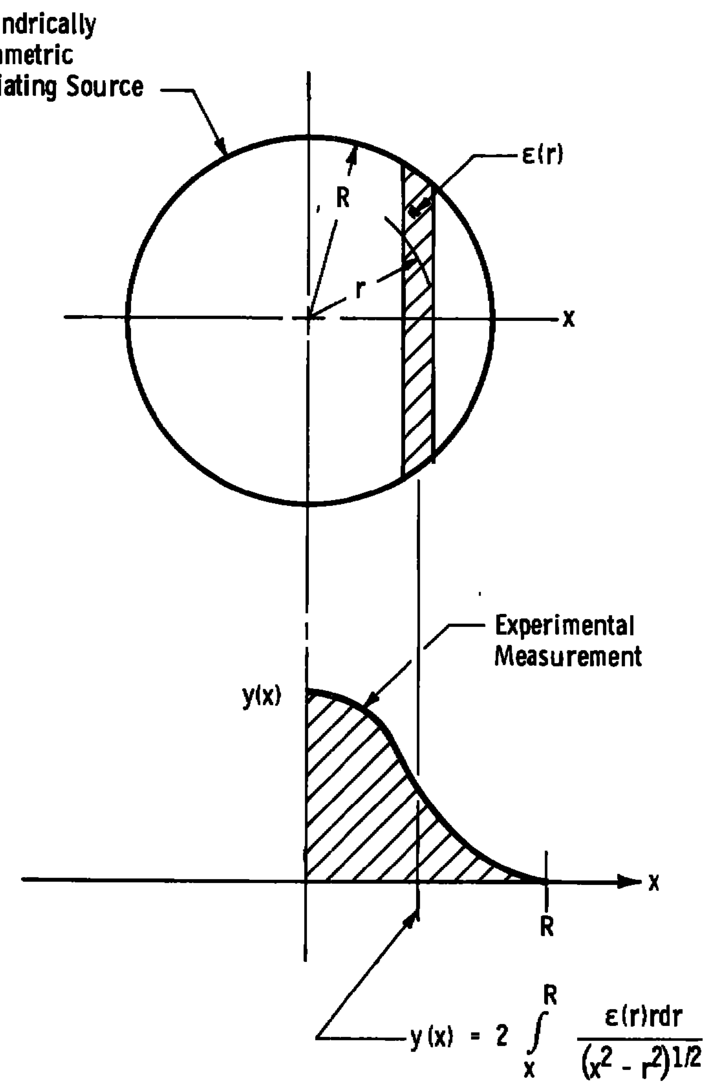


Figure A-1. Illustration of the physical problem.

The smoothing can be accomplished readily by least-squares techniques, and, with the proper choice of fitting function, the determination of the emission coefficient can become analytic. The details of the mathematical development of the equations for smoothing and subsequent emission coefficient and error propagation are given in the body of the report. The problem may be summarized by noting that a set of data points $\{(x, y)\}$, where x is the independent variable and y is the dependent variable, are to be curve fit to a series of sixth-degree even polynomials. Each polynomial is to be valid over a specific range or interval of the data, and adjacent polynomials are to be smooth in both the function and its first derivative; the second derivative is to be continuous at the interval boundaries. The series of polynomials is to be further constrained so that they provide the best least-squares fit to the data. The mathematical problem is solvable by Lagrange's undetermined multipliers. Subsequent to the least-squares curve fitting, the last interval is fitted with a polynomial to insure zero slope and ordinate at the outer edge of the data.

The dependent variable, y, at any point, x, is thus expressible by the equation

$$y(x) = a_{i1}x + a_{i2}x^2 + a_{i3}x^4 + a_{i4}x^6 \tag{A-1}$$

where the subscript i denotes the i^{th} interval such that

$$Z_{i-1} \leq x \leq Z_i \tag{A-2}$$

where the Z's are the interval boundary points. By writing Eq. (A-1) for each of the dependent variables (radiances) measured, one can evolve a system of equations linear in the unknown coefficients, a_{i1}, a_{i2}, a_{i3}, and a_{i4}. The introduction of the constraints provides additional equations also linear in the unknown coefficients and Lagrange's multipliers. The mathematical curve-fitting problem is thus expressible in matrix vector notation as

$$BA = C \tag{A-3}$$

where the matrix A is the (7n - 3) by 1 matrix containing the coefficients of the polynomials and the Lagrangean multipliers and n is the number of intervals into which the data have been divided; B is a (7n - 3) by (7n - 3) symmetric matrix of functions of the independent variable, x, and the interval division points, z; and C is a (7n - 3) by 1 matrix containing functions of the independent and dependent variables. The matrices B and C are described in greater detail below. The objective of the curve fitting technique is to obtain the solution of Eq. (A-3) for the column vector A.

The matrix B may be partitioned thus:

$$B = \begin{bmatrix} R & N \\ N^T & 0 \end{bmatrix} \qquad \text{(A-4)}$$

where R is a 4n by 4n block diagonal matrix and N is a 4n by (3n - 3) block band matrix, each of which may be further partitioned thus:

$$R = \begin{bmatrix} P_1 & & & & \\ & P_2 & & & \\ & & P_3 & & \\ & & & \ddots & \\ & & & & P_n \end{bmatrix} \qquad \text{(A-5)}$$

$$N = \begin{bmatrix} Q_1 & & & & \\ -Q_1 & Q_2 & & & \\ & -Q_2 & Q_3 & \cdot & \\ & & -Q_3 & \ddots & \\ & & & \ddots & Q_{n-1} \\ & & & \cdot & -Q_{n-1} \end{bmatrix} \qquad \text{(A-6)}$$

The matrices P_j are each 4 by 4 symmetric matrices which are defined as

$$P_j = \begin{bmatrix} m_j & \sum_{i=S_j}^{\ell_j} x_i^2 & \sum_{i=S_j}^{\ell_j} x_i^4 & \sum_{i=S_j}^{\ell_j} x_i^6 \\ \sum_{i=S_j}^{\ell_j} x_i^2 & \sum_{i=S_j}^{\ell_j} x_i^4 & \sum_{i=S_j}^{\ell_j} x_i^6 & \sum_{i=S_j}^{\ell_j} x_i^8 \\ \sum_{i=S_j}^{\ell_j} x_i^4 & \sum_{i=S_j}^{\ell_j} x_i^6 & \sum_{i=S_j}^{\ell_j} x_i^8 & \sum_{i=S_j}^{\ell_j} x_i^{10} \\ \sum_{i=S_j}^{\ell_j} x_i^6 & \sum_{i=S_j}^{\ell_j} x_i^8 & \sum_{i=S_j}^{\ell_j} x_i^{10} & \sum_{i=S_j}^{\ell_j} x_i^{12} \end{bmatrix} \tag{A-7}$$

and the matrices Q_j are each 4 by 3 and are defined as

$$Q_j = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{Z_j^2}{2} & Z_j & 1 \\ \frac{Z_j^4}{2} & 2Z_j^3 & 6Z_j^2 \\ \frac{Z_j^6}{2} & 3Z_j^5 & 15Z_j^4 \end{bmatrix} \tag{A-8}$$

The matrix C in Eq. (A-3) is expressed as the product of two matrices,

$$C = GY \tag{A-9}$$

where G is a (7n - 3) by p matrix of certain powers of x (the p-independent data points) and y is a p by 1 matrix of the p-dependent data points. The matrix G may be partitioned into a block diagonal matrix

$$G = \begin{bmatrix} S_1 & & & 0 \\ & S_2 & & \\ & & \ddots & \\ 0 & & & S_n \end{bmatrix} \tag{A-10}$$

where each S_j is a 4 by m_j matrix

$$S_j = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x^2_{m_{j-1}+1} & x^2_{m_{j-1}+2} & \cdots & x^2_{m_{j-1}+m_j} \\ x^4_{m_{j-1}+1} & x^4_{m_{j-1}+2} & \cdots & x^4_{m_{j-1}+m_j} \\ x^6_{m_{j-1}+1} & x^6_{m_{j-1}+2} & \cdots & x^6_{m_{j-1}+m_j} \end{bmatrix} \tag{A-11}$$

With each of the elements of B, G, and Y thus defined for a given set of data (x,y), solution to Eq. (A-3) is immediate, yielding a vector containing the coefficients of the polymonials satisfying the least-squares spline fit criteria. The solution can be expressed as

$$A = WY \tag{A-12}$$

where

$$W = B^{-1} G \tag{A-13}$$

With the dependent data thus expressed as the evaluation of a polynomial, the emission coefficient,

$$\epsilon(r) = -\frac{1}{\pi} \int_r^R \frac{(dy/dx)\,dx}{(x^2 - r^2)^{1/2}} \tag{A-14}$$

is analytic. Since the range of integration is over x, and the dependent variable, y, is expressed by different polynomials over different ranges of x, the integral, Eq. (A-14), is expressed as the sum of integrals, each valid over a different range; i.e.,

$$\epsilon(r) = -\frac{1}{\pi}\left[\int_r^{Z_{j-1}} \frac{dy/dx}{(x^2 - r^2)^{1/2}}\,dx + \sum_{i=j}^{n} \int_{Z_{i-1}}^{Z_i} \frac{dy/dx}{(x^2 - r^2)^{1/2}}\,dx\right] \tag{A-15}$$

Or, substituting the polynomials for the dependent variables,

$$\begin{aligned}\epsilon(r) = &-\frac{(x^2 - r^2)^{1/2}}{\pi}\left[2a_{2,j-1} + \frac{4}{3}a_{3,j-1}(x^2 + 2r^2) + \frac{6}{15}a_{4,j-1}\right.\\ &\left.\cdot\,(3x^4 + 4x^2r^2 + 8r^4)\right]_r^{Z_{j-1}} - \sum_{i=j}^{n}\frac{(x^2 - r^2)^{1/2}}{\pi}\left[2a_{2i}\right.\\ &\left.+\frac{4}{3}a_{3i}(x^2 + 2r^2) + \frac{6}{15}a_{4i}(3x^4 + 4x^2r^2 + 8r^4)\right]\Bigg|_{Z_{i-1}}^{Z_i}\end{aligned} \tag{A-16}$$

which is linear in the polynomial coefficients. Thus, the results of evaluating the emission coefficient at several independent data points can be expressed in matrix vector notation as

$$E = MA \tag{A-17}$$

where E is the column vector of emission coefficients evaluated at m values of r, A is the coefficient matrix determined as the solution to the least-squares spline problem, and M is an m by (7n - 3) matrix defining the coefficients of the respective elements of A.

Substituting for A, the emission coefficient may be expressed as

$$E = MB^{-1}GY \tag{A-18}$$

which expresses the emission coefficient as the result of a linear transformation of the data, y.

With the emission coefficient described as the result of a linear transformation from the data space, it is an easy step to provide the transformation of the uncertainties of the data to uncertainies of the emission coefficient.

Let

$$F = MB^{-1}G \tag{A-19}$$

so that

$$E = FY \tag{A-20}$$

Then

$$[E]_{cv} = F[Y]_{cv}F^T \tag{A-21}$$

where the $[\,]_{cv}$ symbol is used to describe the variance-covariance matrix of the parameter enclosed.

The objective of the computer code presented herein is to calculate the elements of the respective matrices M, B, G, and Y from input data $\{(x, y)\}$ so that the coefficient vector A and the transformation vector F are determined. Further, for input $[Y]_{cv}$, the $[E]_{cv}$ values are determined.

A.1.2 LIMITATIONS

As written, the computer program is subject to the following restrictions and limitations:

1. There can be no more than 51 data points.

2. There can be no more than 10 intervals.

3. There must be at least one point per interval.

4. There must be at least two intervals.

5. The variance-covariance matrix of the raw data must be diagonal.

6. The input data must be read in by increasing displacement.

A.2.0 PROGRAM DESCRIPTION

The program in its present form requires approximately 150 K bytes of core on the IBM 370/165 computer, is composed of six Fortran subroutines or functions and a main program, and will perform the computations for 31 data points distributed into four intervals in about 1/2 sec. The time required per case, of course, varies according to the number of points, intervals, and points per interval.

A.2.1 SUBPROGRAM DESCRIPTIONS

A short description of each pertinent routine used in the computer program is listed below.

MAIN PROGRAM

The main program provides the logic for the computer program to execute multiple cases, the proper calling logic to the various routines which effect the data input, inversion, and errors propagation analysis, and the program summary output. The program utilizes two output units:

the line printer and logical unit 8. The logical unit 8 provides an additional temporary storage device so that the results of the analysis may be used in subsequent analysis programs by job stepping. Multiple data cases are accomplished simply by putting the proper logic inside a Do-loop.

SUBROUTINE INPUT

As the name implies, this subroutine provides for the input of all data on logical unit 5. The subroutine includes calibration calculations to provide for conversion of the input raw data units to physical units. The input data in raw and calibrated form are output on logical unit 6. All communication with this subrountine is through COMMON.

SUBROUTINE INVERT

This subroutine provides the logic to perform the least-squares spline fitting of the data and determination of the coefficients and transformation matrix. The bulk of this work is accomplished in SUBROUTINE COVCAL. However, subsequent to the call to COVCAL, the subroutine recalculates the coefficients for the last interval in order to assure meeting the constraints of zero slope and ordinate at the outer edge of the data. The final set of calculated coefficients are output on logical unit 6, and the percentage difference between the input data and the results of the curve fit are calculated. All communication with this subroutine is through COMMON.

SUBROUTINE COVCAL

This subroutine provides for the determination of the elements of the matrix B [Eqs. (A-3), (A-4), and (A-5) through (A-8)], the matrix G [Eqs. (A-9) through (A-11)], the matrix M [Eq. (A-17)] and the matrix

F [Eq. (A-19)]. With these matrices, the polynomial coefficients, the emission coefficients, and the propagated variance-covariance matrix are immediate. The calculations proceed calculating sequentially as follows:

1. The array G [Eq. (A-9) through (A-11)], identified in Fortran as XT

2. The array B and B^{-1} [Eqs. (A-3) through (A-8)], identified in Fortran as (XTX)

3. The coefficients array A [(Eq. (A-12)], identified in Fortran as AV

4. The array M [(Eq. (A-17)], identified in Fortran as XTX)

5. The array F [(Eq. (A-19)], identified in Fortran as XT)

6. The array $[E]_{cv}$ [(Eq. (A-21)], identified in Fortran as VC.

All communication with this subroutine is through COMMON.

SUBROUTINE EMSCAL

This subroutine provides the logic for calculation of the emission coefficients at each of the radial positions, Eq. (A-16), numerically equal to the input displacement positions. The calculation proceeds by Do-loop, and tracking of the point with respect to the interval boundary points is maintained so that the correct coefficients are used in the integral evaluation. Communication with this subroutine is through the argument list: No. of points, No. of intervals, interval endpoints, displacement array, curve fit coefficient arrays, and resultant emission coefficient array.

FUNCTION EMFUN

This double precision function performs the numerical evaluation of the Abel integral at either the upper or lower limit when the intensity is described by a four-term, sixth-degree even polynomial, Eq. (A-16). Input arguments include the radius, the upper weighted by the exponent of the corresponding independent variable (that is, $2a_2$, $4a_3$, $6a_4$). These weighting factors arise from the differentiation of the polynomial.

SUBROUTINE MATINV

This is a standard service routine which obtains the inverse of a matrix by Crout reduction with partial pivoting.

A.2.2 LIST OF FORTRAN VARIABLES

A	Spontaneous transition probability (sec^{-1})
A1(I)	Curve fit coefficient for the I^{th} interval, Eq. (A-3)
A2(I)	Curve fit coefficient for the I^{th} interval, Eq. (A-3)
A3(I)	Curve fit coefficient for the I^{th} interval, Eq. (A-3)
A4(I)	Curve fit coefficient for the I^{th} interval, Eq. (A-3)
C1	Recalculated coefficient for last interval: A1(NTVL)
C2	Recalculated coefficient for last interval: A2(NTVL)

C3	Recalculated coefficient for last interval: A3(NTVL)
C4	Recalculated coefficient for last interval: A4(NTVL)
CAL2	Calibration constant for intensity according to: $ENTEN_{cal} = (ENTEN_{input} - ZERO) *CAL2$
CAL3 CAL4	Calibration constant for x according to: $DISP_{cal} = (DISP_{input} - CAL3) * CAL4$
CAL5 CAL6	Not used
D	Intermediate computational variable
DEN	Intermediate computational variable
DISP	Array of displacements at which data are taken
EMIS	Vector E of emission coefficients
EN	Energy level (cm^{-1}), not used
ENDPT	Array of interval endpoints
ENTEN	Vector Y of radiances at the x locations in DISP
F	Intermediate variable
F1	Assignment statement subprograms for the computation of Eq. (A-16), Subroutine COVCAL

F2	Assignment statement subprograms for the computation of Eq. (A-16), Subroutine COVCAL
F3	Assignment statement subprograms for the computation of Eq. (A-16), Subroutine COVCAL
FAC	Fractional value of input radiance data which is to be taken as the standard deviation of the radiance
G	Statistical weight
HEAD	Alphanumeric header for identification
I	Index
IBOT	Index
ITIME	Code to indicate whether x locations and interval sizes are the same from one data set to the next
ITOP	NTVL + 1
J	Index
J1	Index
J2	Index
J24	Index
JC	Index
K	Index

K1	Index
K2	Index
K3	Index
K4	Index
L	Index
LIN	Line Counter index
M2	Index
M3	Index
MD	7*NTVL - 3 = 7n - 3
MSV	Intermediate variable
NPNTVL	Number of points in the I^{th} interval, m_i
NPT	Total number of data points, p
NPT1	Computational variable
NSETS	Number of sets of data
NTLO	Computational variable
NTVL	Number of intervals, n
NTVL4	Computational variable

NTVL41	Computational variable
NTVL42	Computational variable
PERROR	Percentage error between curve fit radiances and input data values
PI	3.14159265
R	A particular x value passed to EMFUN from subroutine EMSCAL
RO	Overall radius of source, R_o
SD	Array of intensity data standard deviations
SUM	Intermediate computational variable
T1	Computational variable used to compute Eq. (A-16)
T2	Computational variable used to compute Eq. (A-16)
T3	Computational variable used to compute Eq. (A-16)
VC	Emission coefficient covariance matrix $[E]_{cv}$
WL	Wavelength (angstroms), not used
WORK	Array containing the squares of the interval end points

X	Intermediate computational variable in subroutines INVERT and EMSCAL: array used first to store the elements of the matrix W, and next the elements of the matrix F $[Y]_{cv}$ in subroutine COVCAL
XT	Array containing the elements of the matrix G, and then the elements of the matrix F
XTX	Array containing the elements of the matrix B, next the elements of the matrix B^{-1}, and finally the elements of the matrix M
Y	A particular endpoint passed to EMFUN from EMSCAL
Y1	Intermediate variable
Y1P	Array of curve fit intensity values
YSPLN	Array containing data intensity vaues
Z1	Intermediate variable
Z2	Intermediate variable
Z12	Intermediate variable
Z13	Intermediate variable
Z21	Intermediate variable
Z22	Intermediate variable

ZERO Calibration constant for radiance according to:

$$ENTEN_{cal} = (ENTEN_{input} - ZERO)\ \ CAL2$$

ZSPLN Array containing squared x values

A.3.0 INPUT/OUTPUT

The input consists of certain control parameters defining the logical arrangement of the physical data, the radiance data and the corresponding position (measured from zero on the centerline) and standard deviation estimate, and calibration factors. In addition, there are four input parameters which are not used in the inversion but are passed on to the output data unit for subsequent use. When narrow spectral line data are inverted, the four additional parameters can be used for the atomic constants characterizing the radiation. The output consists of input data, calibrated data, certain intermediate calculation steps, and final results. The output is clearly labeled.

The principal input physical parameter is the radiance (radiated power per unit area per unit solid angle). The principal output physical parameter is the emission coefficient (radiated power per unit volume per unit solid angle). The specific units of the calculated emission coefficient will be consistent with the units chosen for the input radiance and displacement. There is no internal unit conversion provided other than with the calibration factors.

A.3.1 INPUT DATA CARDS

Card No. and Format	Fortran Variable	Description
1. (I3)	NSETS	Number of sets of data to be radially inverted.

2.	(20A4)	HEAD	Header card to provide means of identifying uniquely each set of input data.
3.	(6E12.0)	WL,A,G,EN	Four variables not used directly in the calculation but passed through to the output unit for subsequent use. For narrow spectral line emission data these may be wavelength, transition probability, statistical weight, and energy level respectively.
4.	(3I3)	NPT	Number of data points.
		NTVL	Number of intervals.
		ITIME	= 0 if only the radiances have changed from the previous data set. = 1 each time a new set of data is run.
5.	(26I3)	(NPNTVL(I),I=1,NTVL)	NTVL values: each value is the number of data points in the corresponding interval.

6. (6E12.0) ↓ n.	DISP(I) ENTEN(I) SD (I) I = 1, NPT	The displacement radiance and standard deviation data from the centerline to the outer edge, two sets per card.
n+1. (6E12.0)	ZERO CAL2	Radiance data calibration according to I = (ENTEN - ZERO)*CAL2.
	CAL3 CAL4	Displacement data calibration according to X = (DISP - CAL3) *CAL4.
	CAL5	Not used.
	CAL6	Not used.

For multiple data sets, repeat cards 2 through n+1.

A.3.2 OUTPUT

The printed output, on logical unit 6, consists of five pages/case, and the identification and logic of the output are self-evident. The printed output consists of 1) input data, 2) calibration data, 3) curve fit coefficients, 4) emission coefficient, and 5) propagated errors.

In addition to the printed output, the various quantities are written (formatted) on logical unit 8 for offline storage of results or use by subsequent job steps. This output is listed as follows:

Record No. and Format	Output List	Description
1. (I3)	J	Case No.

2.	(20A4)	HEAD	Case identification.
3.	(4D20.13)	WL,A,G,EN	Four variables passed from input to this output record; not used in the radial inversion.
4.	(26I3)	NPT, NTVL, (NPNTVL,I = 1, NTVL)	No. of points, No. of intervals and No. of points in each interval.
5.	(4D20.13)	(A1(I),A2(I),A3(I), A4(I),I = 1, NTVL)	The curve fit coefficients for each of the intervals.
6.	(4D20.13)	(ENDPT(I),I = 1,NTVL+1)	The endpoints of each interval.
7.	(4D20.13)	(DISP(I),ENTEN(I),I = 1,NPT)	Ordered pairs of displacement and radiance, two pairs to each record.
8.	(4D20.13)	(PERRØR(I),I = 1,NPT)	Percentage error between input and fitted radiances.
9.	(4D20.13)	(EMIS(I),I = 1,NPT)	Calculated emission coefficients.

10. (4D20.13) (SD(I),I = 1,NPT) Standard deviation of input radiances.

11. (4D20.13) (VCLI),I = 1,NPT) Emission coefficient standard deviations.

A.3.3 SAMPLE INPUT SHEET

ARO CARD FORMAT

JOB TITLE	PROJECT NO.	PROGRAMMER	PAGE OF	DATE

```
  1
CHECKØUT DATA USING DEXP(-X*X) FØR INPUT DATA

 0.0
 31  4

  7  8  8  8
 0.0        1.0         0.01        0.1         0.99005     0.099

0.2         0.96079     0.0961      0.3         0.91393     0.0914
0.4         0.85214     0.0852      0.5         0.7788      0.0779
0.6         0.69768     0.0698      0.7         0.61263     0.0613
0.8         0.52729     0.0527      0.9         0.4486      0.0449

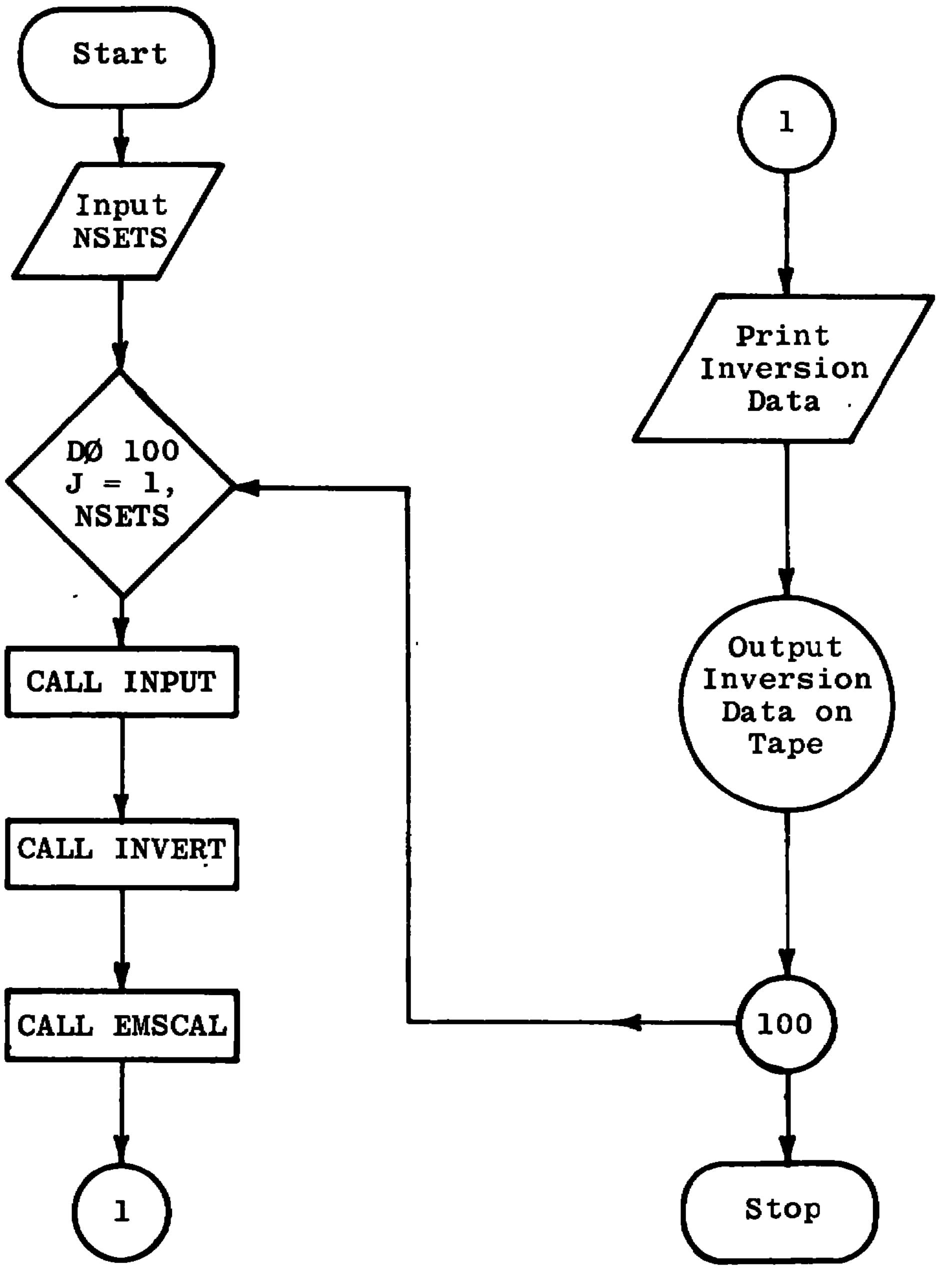
1.0         0.36788     0.0368      1.1         0.2982      0.0298
1.2         0.23693     0.0237      1.3         0.18453     0.0184

1.4         0.14086     0.0141      1.5         0.1054      0.0105
1.6         0.077305    0.0077      1.7         0.055576    0.0056

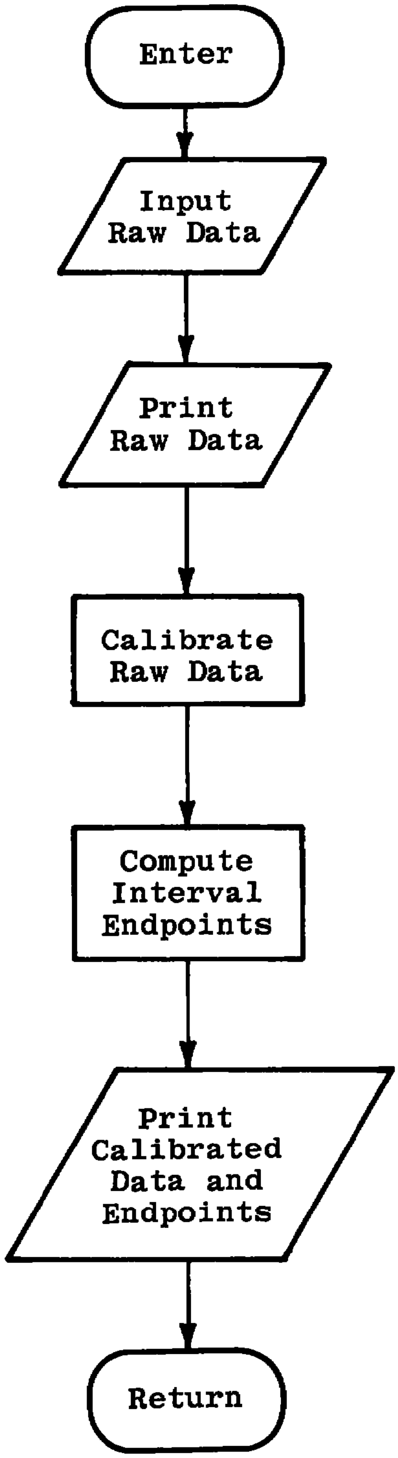
1.8         0.039164    0.0039      1.9         0.027052    0.0027
2.0         0.018316    0.0018      2.1         0.012155    0.0012
2.2         7.9071    -03 .791    -042.3        5.0418    -030.504     -04
2.4         3.1511    -03 .315    -042.5        1.9305    -030.193     -04
  2.6       1.1592  -03   .116    -042.7        6.8233    -040.682     -05
  2.8       3.9367  -04   .394    -052.9        2.226     -040.223     -05

  3.0       1.2341  -04   .123    -05
  0.0       1.0           0.0         1.0
```

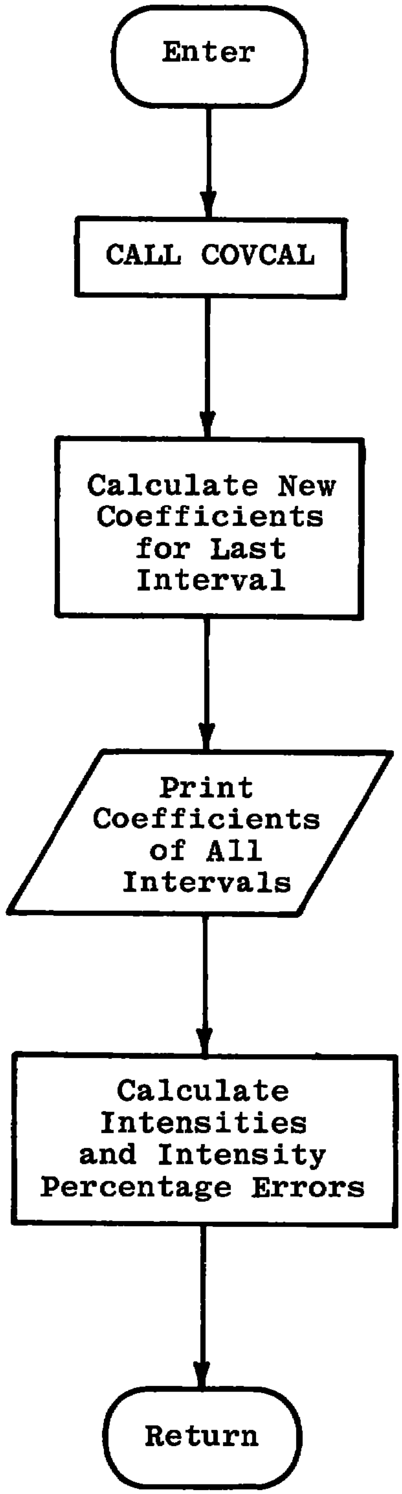
A.4.0 FLOW CHARTS



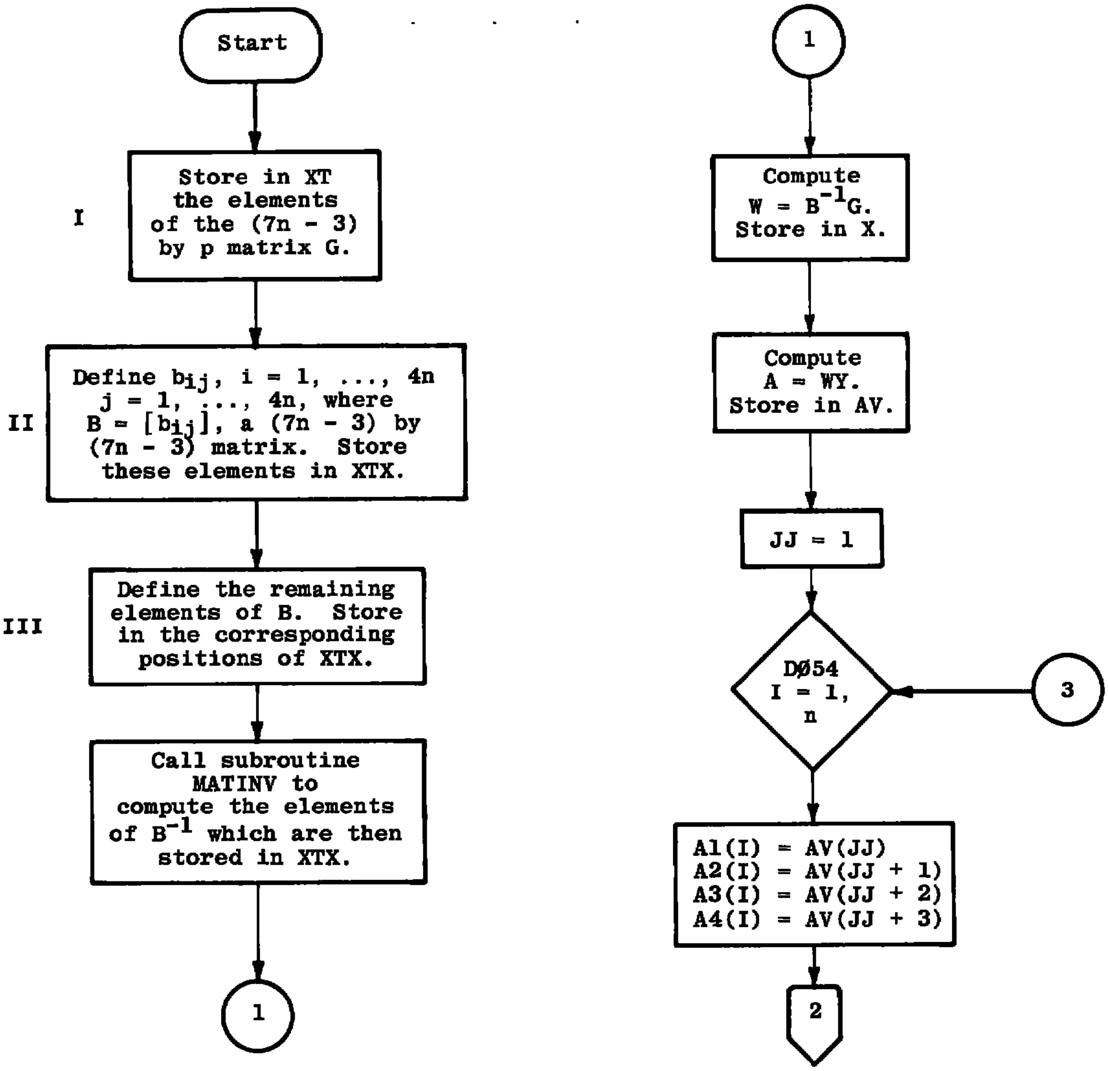
MAIN PROGRAM



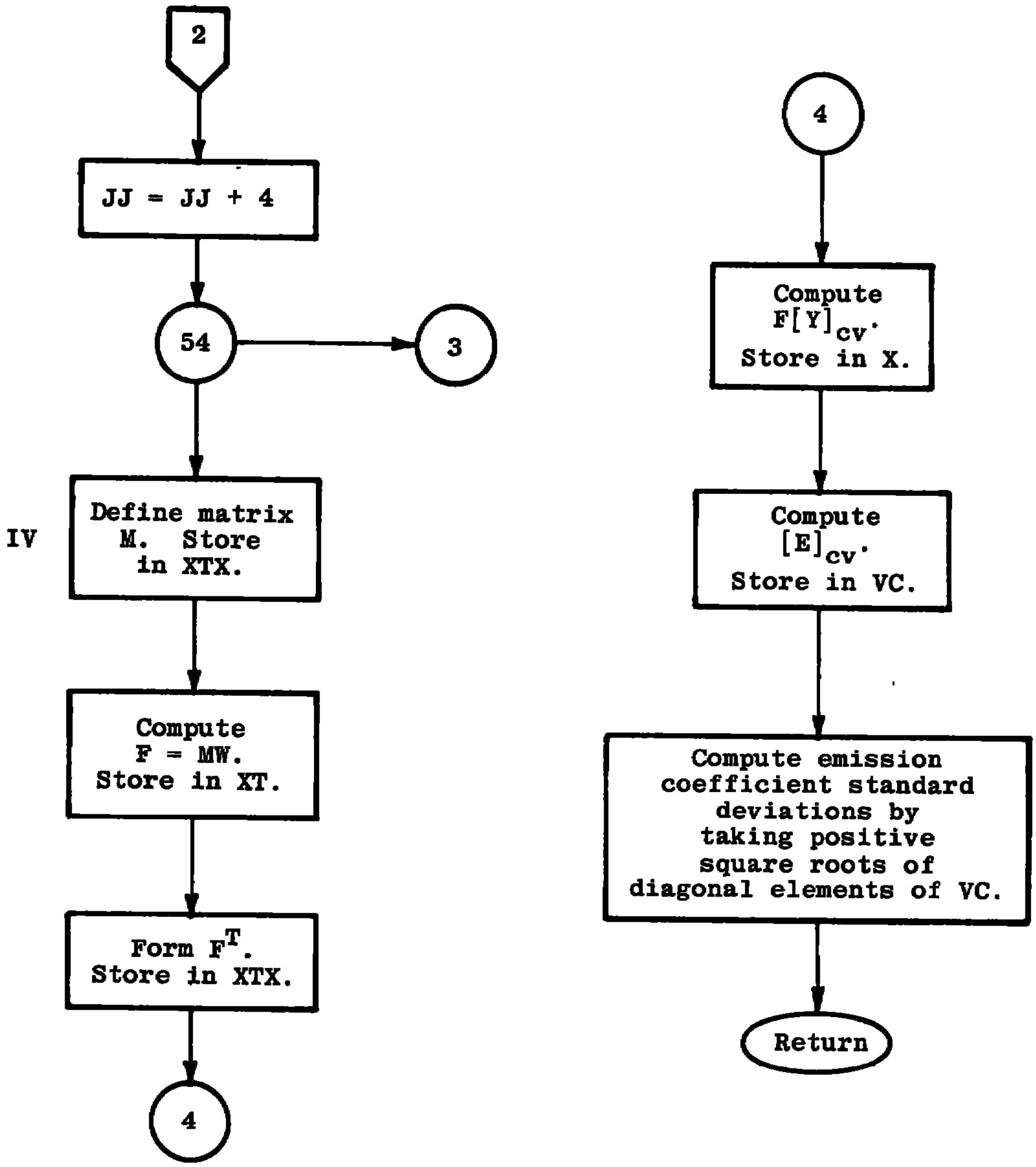
SUBROUTINE INPUT



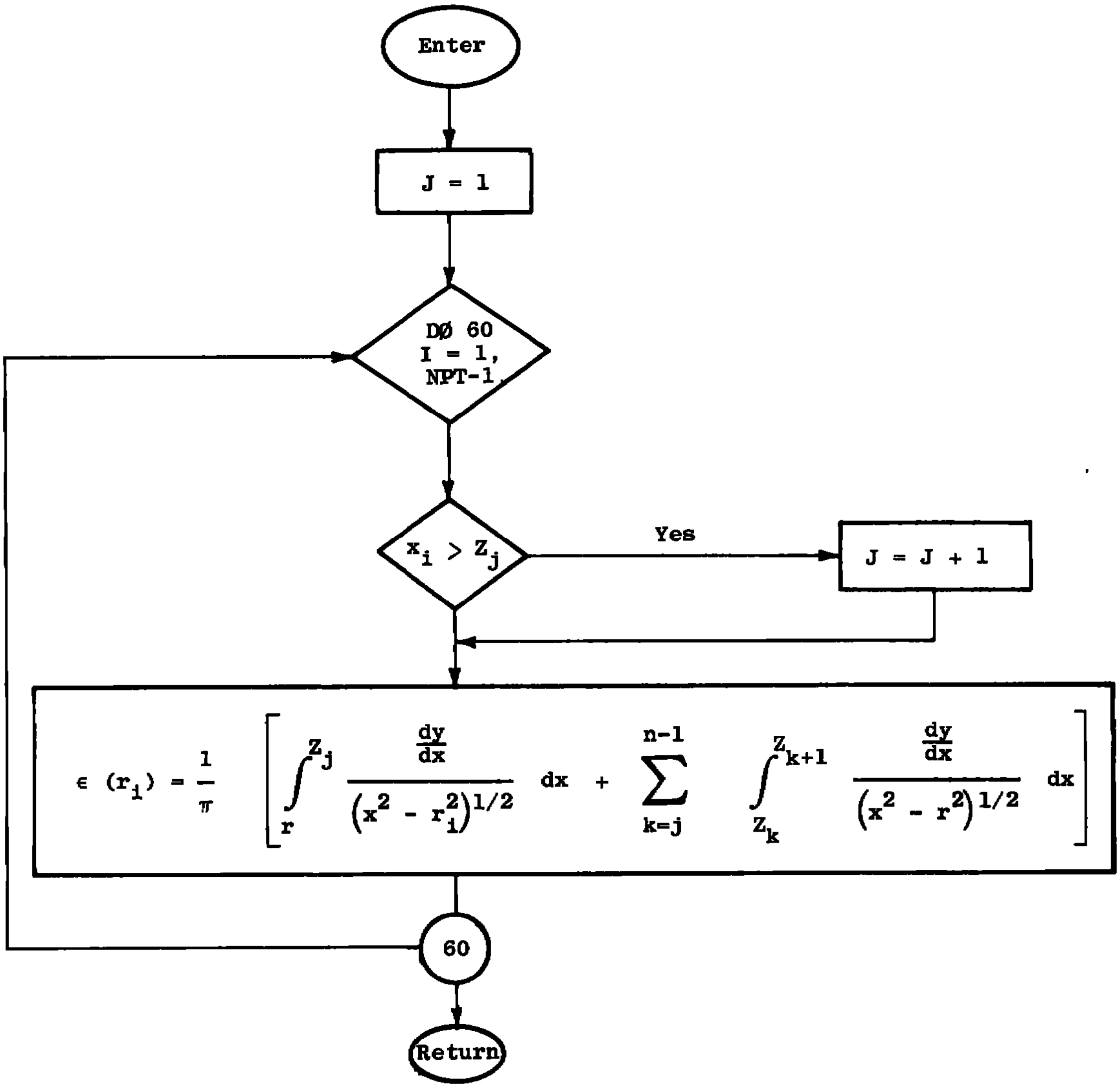
SUBROUTINE INVERT



SUBROUTINE COVCAL



SUBROUTINE COVCAL, CONCLUDED



Note: In this subroutine, the subscript j (denoting interval endpoint) is one digit larger than the subscript i in Eq. (A-2).

SUBROUTINE EMSCAL

Enter

Evaluate the term

$$\rho = \frac{\sqrt{x^2 - r^2}}{\pi} \left[2a_2 + \frac{4}{3} a_3 \left(x^2 + 2r^2\right) + \frac{6}{15} a_4 \left(3x^4 + 4x^2r^2 + 8r^4\right)\right]$$

used in Eq. (A-16)

Return

FUNCTION EMFUN

A.5.0 OUTPUT

```
 CHECKOUT DATA USING DEXP(-X*X) FOR INPUT DATA
INPUT DATA

NUMBER OF POINTS= 31
NUMBER OF INTERVALS=  4
NUMBER OF POINTS PER INTERVAL:  7  8  8  8

INPUT DATA ARRAY
 0.0                1.000000D 00         1.000000D-02
1.000000D-01       9.900500D-01         9.900000D-02
2.000000D-01       9.607900D-01         9.610000D-02
3.000000D-01       9.139300D-01         9.140000D-02
4.000000D-01       8.521400D-01         8.520000D-02
5.000000D-01       7.788000D-01         7.790000D-02
6.000000D-01       6.976800D-01         6.980000D-02
7.000000D-01       6.126300D-01         6.130000D-02
8.000000D-01       5.272900D-01         5.270000D-02
9.000000D-01       4.486000D-01         4.490000D-02
1.000000D 00       3.678800D-01         3.680000D-02
1.100000D 00       2.982000D-01         2.980000D-02
1.200000D 00       2.369300D-01         2.370000D-02
1.300000D 00       1.845300D-01         1.840000D-02
1.400000D 00       1.408600D-01         1.410000D-02
1.500000D 00       9.105400D 10         1.050000D-02
1.600000D 00       7.730500D-02         7.700000D-03
1.700000D 00       5.557600D-02         5.600000D-03
1.800000D 00       3.916400D-02         3.900000D-03
1.900000D 00       2.705200D-02         2.700000D-03
2.000000D 00       1.831600D-02         1.800000D-03
2.100000D 00       1.215500D-02         1.200000D-03
2.200000D 00       7.907100D-03         7.910000D-05
2.300000D 00       5.041800D-03         5.040000D-05
2.400000D 00       3.151100D-03         3.150000D-05
2.500000D 00       1.930500D-03         1.930000D-05
2.600000D 00       1.159200D-03         1.160000D-05
2.700000D 00       6.823300D-04         6.820000D-06
2.800000D 00       3.936700D-04         3.940000D-06
2.900000D 00       2.226000D-04         2.230000D-06
3.000000D 00       1.234100D-04         1.230000D-06

BEGINNING POINT OF EACH INTERVAL:
 0.0           6.500000D-01 1.450000D 00 2.250000D 00
```

CHECKOUT DATA USING DEXP(-X*X) FOR INPUT DATA
INPUT (X,INTENSITY,STD DEV) ARRAY

0.0	1.000000D 00	1.000000D-02
1.000000D-01	9.900500D-01	9.900000D-02
2.000000D-01	9.607900D-01	9.610000D-02
3.000000D-01	9.139300D-01	9.140000D-02
4.000000D-01	8.521400D-01	8.520000D-02
5.000000D-01	7.788000D-01	7.790000D-02
6.000000D-01	6.976800D-01	6.980000D-02
7.000000D-01	6.126300D-01	6.130000D-02
8.000000D-01	5.272900D-01	5.270000D-02
9.000000D-01	4.486000D-01	4.490000D-02
1.000000D 00	3.678800D-01	3.680000D-02
1.100000D 00	2.982000D-01	2.980000D-02
1.200000D 00	2.369300D-01	2.370000D-02
1.300000D 00	1.845300D-01	1.840000D-02
1.400000D 00	1.408600D-01	1.410000D-02
1.500000D 00	9.105400D 10	1.050000D-02
1.600000D 00	7.730500D-02	7.700000D-03
1.700000D 00	5.557600D-02	5.600000D-03
1.800000D 00	3.916400D-02	3.900000D-03
1.900000D 00	2.705200D-02	2.700000D-03
2.000000D 00	1.831600D-02	1.800000D-03
2.100000D 00	1.215500D-02	1.200000D-03
2.200000D 00	7.907100D-03	7.910000D-05
2.300000D 00	5.041800D-03	5.040000D-05
2.400000D 00	3.151100D-03	3.150000D-05
2.500000D 00	1.930500D-03	1.930000D-05
2.600000D 00	1.159200D-03	1.160000D-05
2.700000D 00	6.823300D-04	6.820000D-06
2.800000D 00	3.936700D-04	3.940000D-06
2.900000D 00	2.226000D-04	2.230000D-06
3.000000D 00	1.234100D-04	1.230000D-06

CHECKOUT DATA USING DEXP(-X*X) FOR INPUT DATA
CUBIC COEFFICIENTS

INTERVAL START	INTERVAL END	A1	A2	A3	A4
0.0	6.500000D-01	-2.341060D 08	1.077542D 10	-5.172606D 10	5.391526D 10
6.500000D-01	1.450000D 00	4.229481D 09	-2.091869D 10	2.328958D 10	-5.268680D 09
1.450000D 00	2.250000D 00	-6.156878D 10	7.296705D 10	-2.136476D 10	1.810882D 09
2.250000D 00	3.000000D 00	1.481143D 11	-6.709735D 10	9.424808D 09	-4.220132D 08

CHECKOUT DATA USING DEXP(-X*X) FOR INPUT DATA
INVERSION RESULTS

STA.	DISPLACEMENT	INTENSITY(DATA)	INTENSITY(CALC)	PERCENT ERROR	EMISSION COEF
1	0.0	1.000000D 00	-2.341060D 08	-2.341060D 10	-2.419098D 09
2	1.000000D-01	9.900500D-01	-1.314705D 08	-1.327918D 10	-2.217571D 09
3	2.000000D-01	9.607900D-01	1.175995D 08	1.223987D 10	-1.684278D 09
4	3.000000D-01	9.139300D-01	3.560045D 08	3.895314D 10	-1.022644D 09
5	4.000000D-01	8.521400D-01	3.866101D 08	4.536932D 10	-5.347347D 08
6	5.000000D-01	7.788000D-01	6.929476D 07	8.897632D 09	-5.560432D 08
7	6.000000D-01	6.976800D-01	-5.431840D 08	-7.785576D 10	-1.330366D 09
8	7.000000D-01	6.126300D-01	-1.048706D 09	-1.711810D 11	-2.648561D 09
9	8.000000D-01	5.272900D-01	-1.000224D 09	-1.896915D 11	-3.791463D 09
10	9.000000D-01	4.486000D-01	-2.343609D 08	-5.224272D 10	-4.546320D 09
11	1.000000D 00	3.678800D-01	1.331686D 09	3.619892D 11	-4.767014D 09
12	1.100000D 00	2.982000D-01	3.682346D 09	1.234858D 12	-4.329970D 09
13	1.200000D 00	2.369300D-01	6.667638D 09	2.814181D 12	-3.155299D 09
14	1.300000D 00	1.845300D-01	9.963342D 09	5.399307D 12	-1.240086D 09
15	1.400000D 00	1.408600D-01	1.302737D 10	9.248452D 12	1.277675D 09
16	1.500000D 00	9.105400D 10	1.507507D 10	-8.344381D 01	3.895904D 09
17	1.600000D 00	7.730500D-02	1.559234D 10	2.016990D 13	5.880261D 09
18	1.700000D 00	5.557600D-02	1.457568D 10	2.622658D 13	7.082775D 09
19	1.800000D 00	3.916400D-02	1.215789D 10	3.104354D 13	7.439705D 09
20	1.900000D 00	2.705200D-02	8.609133D 09	3.182439D 13	6.929723D 09
21	2.000000D 00	1.831600D-02	4.359721D 09	2.380280D 13	5.594712D 09
22	2.100000D 00	1.215500D-02	2.426017D 07	1.995901D 11	3.577204D 09
23	2.200000D 00	7.907100D-03	-3.572939D 09	-4.518646D 13	1.212775D 09
24	2.300000D 00	5.041800D-03	-5.559027D 09	-1.102588D 14	-5.504301D 08
25	2.400000D 00	3.151100D-03	-6.321916D 09	-2.006257D 14	-1.627481D 09
26	2.500000D 00	1.930500D-03	-6.118149D 09	-3.169204D 14	-2.233169D 09
27	2.600000D 00	1.159200D-03	-5.139217D 09	-4.433417D 14	-2.376671D 09
28	2.700000D 00	6.823300D-04	-3.648999D 09	-5.347851D 14	-2.094670D 09
29	2.800000D 00	3.936700D-04	-1.991505D 09	-5.058817D 14	-1.465707D 09
30	2.900000D 00	2.226000D-04	-5.989269D 08	-2.690597D 14	-6.427049D 08
31	3.000000D 00	1.234100D-04	0.0	-1.000000D 02	0.0

CHECKOUT DATA USING DEXP(-X*X) FOR INPUT DATA
ERROR ANALYSIS RESULTS: STANDARD DEVIATION

DISPLACEMENT	INTENSITY(DATA)	STD DEV(INTENSITY)	EMIS COEF	STD DEV(EMIS COEF)
0.0	1.0000000D 00	1.000000D-02	-2.419098D 09	1.007021D-01
1.000000D-01	9.900500D-01	9.900000D-02	-2.217571D 09	9.297617D-02
2.000000D-01	9.607900D-01	9.610000D-02	-1.684278D 09	7.192315D-02
3.000000D-01	9.139300D-01	9.140000D-02	-1.022644D 09	4.437996D-02
4.000000D-01	8.521400D-01	8.520000D-02	-5.347347D 08	2.604675D-02
5.000000D-01	7.788000D-01	7.790000D-02	-5.560432D 08	3.148035D-02
6.000000D-01	6.976800D-01	6.980000D-02	-1.330366D 09	3.490664D-02
7.000000D-01	6.126300D-01	6.130000D-02	-2.648561D 09	2.710171D-02
8.000000D-01	5.272900D-01	5.270000D-02	-3.791463D 09	1.868135D-02
9.000000D-01	4.486000D-01	4.490000D-02	-4.546320D 09	1.186654D-02
1.000000D 00	3.678800D-01	3.680000D-02	-4.767014D 09	8.482132D-03
1.100000D 00	2.982000D-01	2.980000D-02	-4.329970D 09	8.980274D-03
1.200000D 00	2.369300D-01	2.370000D-02	-3.155299D 09	1.012432D-02
1.300000D 00	1.845300D-01	1.840000D-02	-1.240086D 09	9.731331D-03
1.400000D 00	1.408600D-01	1.410000D-02	1.277675D 09	7.369577D-03
1.500000D 00	9.105400D 10	1.050000D-02	3.895904D 09	4.059518D-03
1.600000D 00	7.730500D-02	7.700000D-03	5.880261D 09	2.051581D-03
1.700000D 00	5.557600D-02	5.600000D-03	7.082775D 09	2.641147D-03
1.800000D 00	3.916400D-02	3.900000D-03	7.439705D 09	3.706994D-03
1.900000D 00	2.705200D-02	2.700000D-03	6.929723D 09	4.110786D-03
2.000000D 00	1.831600D-02	1.800000D-03	5.594712D 09	3.697116D-03
2.100000D 00	1.215500D-02	1.200000D-03	3.577204D 09	2.522184D-03
2.200000D 00	7.907100D-03	7.910000D-05	1.212775D 09	8.340273D-04
2.300000D 00	5.041800D-03	5.040000D-05	-5.504301D 08	9.446839D-04
2.400000D 00	3.151100D-03	3.150000D-05	-1.627481D 09	1.772534D-03
2.500000D 00	1.930500D-03	1.930000D-05	-2.233169D 09	1.763456D-03
2.600000D 00	1.159200D-03	1.160000D-05	-2.376671D 09	9.918492D-04
2.700000D 00	6.823300D-04	6.820000D-06	-2.094670D 09	3.735650D-04
2.800000D 00	3.936700D-04	3.940000D-06	-1.465707D 09	1.962961D-03
2.900000D 00	2.226000D-04	2.230000D-06	-6.427049D 08	3.092819D-03
3.000000D 00	1.234100D-04	1.230000D-06	0.0	0.0

A.6.0 FORTRAN LISTING

```
LEVEL 21.7 ( JAN 73 )                    OS/360  FORTRAN H                              DATE  76.271/08.51.03

          COMPILER OPTIONS - NAME=  MAIN,OPT=02,LINECNT=58,SIZE=0000K,
                             SOURCE,EBCDIC,NOLIST,NODECK,LOAD,MAP,NOEDIT,ID,XREF
                 C      THIS PROGRAM PERFORMS RADIAL INVERSIONS BY LEAST SQUARES SPLINE
                 C      FITTING THE RAW DATA AND THEN INVERTING THE RAW DATA
                 C
  ISN 0002              IMPLICIT REAL*8(A-H,O-Z)
  ISN 0003              INTEGER HEAD(20)
  ISN 0004              DIMENSION NPNTVL(10),DISP(51),ENTEN(51),ENDPT(12),A1(10),A2(10),
                       1 A3(10),A4(10),EMIS(51),YCALC(51),PERROR(51),SD(51),VC(51,51)
  ISN 0005              COMMON DISP,ENTEN,ENDPT,RO,A1,A2,A3,A4,YCALC,PERROR,EMIS,HEAD,
                       1 WL,A,G,FN,                        SD,VC
  ISN 0006              COMMON NPT,NPNTVL,NTVL
                 C
                 C
  ISN 0007              READ(5,1005)NSETS
  ISN 0008              DO 100 J=1,NSETS
                 C
  ISN 0009              CALL INPUT
  ISN 0010              IF(J.EQ.1)ITIME=1
                 C
                 C
  ISN 0012              CALL INVERT
                 C
                 C
  ISN 0013              CALL EMSCAL(NPT,NTVL,ENDPT,DISP ,A1(1),A2(1),A3(1),A4(1),EMIS)
                 C
  ISN 0014              LIN=50
  ISN 0015              DO 30 I=1,NPT
  ISN 0016              IF(LIN.LT.50)GO TO 28
  ISN 0018              WRITE(6,1000)HEAD
  ISN 0019              WRITE(6,1003)
  ISN 0020              LIN=6
  ISN 0021           28 CONTINUE
  ISN 0022              WRITE(6,1004)I,DISP(I),ENTEN(I),YCALC(I),PERROR(I),EMIS(I)
  ISN 0023              LIN=LIN+1
  ISN 0024           30 CONTINUE
  ISN 0025              WRITE(8,2000)J
  ISN 0026              WRITE(8,2001)HEAD
  ISN 0027              WRITE(8,2002)WL,A,G,EN
  ISN 0028              WRITE(8,2000)NPT,NTVL,(NPNTVL(I),I=1,NTVL)
  ISN 0029              ITOP=NTVL+1
  ISN 0030              WRITE(8,2002)(A1(I),A2(I),A3(I),A4(I),I=1,NTVL)
  ISN 0031              WRITE(8,2002)(ENDPT(I),I=1,ITOP)
  ISN 0032              WRITE(8,2002)(DISP(I),ENTEN(I),I=1,NPT)
  ISN 0033              WRITE(8,2002)(PERROR(I),I=1,NPT)
  ISN 0034              WRITE(8,2002)(EMIS(I),I=1,NPT)
  ISN 0035              WRITE(8,2002)(SD(I),I=1,NPT)
  ISN 0036              WRITE(8,2002)(VC(I,I),I=1,NPT)
  ISN 0037              WRITE(6,1000)HEAD
  ISN 0038              WRITE(6,3000)
  ISN 0039              WRITE(6,3002)
  ISN 0040              WRITE(6,3001)(DISP(I),ENTEN(I),SD(I),EMIS(I),VC(I,I),I=1,NPT)
  ISN 0041          100 CONTINUE
  ISN 0042              RETURN
                 C
  ISN 0043         1000 FORMAT ('1',20A4)
```

```
                C
ISN 0044         1003 FORMAT(1X,'INVERSION RESULTS',///1X,'STA.',5X,'DISPLACEMENT',
                     2 5X,'INTENSITY(DATA)',5X,'INTENSITY(CALC)',5X,'PERCENT ERROR',
                     3 5X,'EMISSION COEF')
                C
ISN 0045         1004 FORMAT(1X,I3,1P5E19.6)
                C
ISN 0046         1005 FORMAT(I3)
                C
ISN 0047         2000 FORMAT(26I3)
                C
ISN 0048         2001 FORMAT(20A4)
                C
ISN 0049         2002 FORMAT(4D20.13)
                C
ISN 0050         3000 FORMAT(1X,'ERROR ANALYSIS RESULTS:  STANDARD DEVIATION',///)
ISN 0051         3001 FORMAT(1X,1P5E19.6)
ISN 0052         3002 FORMAT(8X,'DISPLACEMENT',6X,'INTENSITY(DATA)',2X,'STD DEV(INTENSIT
                     1Y)',5X,'EMIS COEF',6X,'STD DEV(EMIS COEF)')
ISN 0053              END
```

```
LEVEL 21.7 ( JAN 73 )                    OS/360  FORTRAN H                                  DATE  76.271/08.51.06

          COMPILER OPTIONS - NAME=  MAIN,OPT=02,LINECNT=58,SIZE=0000K,
                             SOURCE,EBCDIC,NOLIST,NODECK,LOAD,MAP,NOEDIT,ID,XREF
  ISN 0002                SUBROUTINE INPUT
                 C
                 C
                 C        THIS SUBROUTINE DOES THE INPUT OF DATA FOR ABEL INVERSION
                 C
                 C
  ISN 0003                IMPLICIT REAL*8(A-H,O-Z)
  ISN 0004                INTEGER HEAD(20)
                 C
  ISN 0005                DIMENSION NPNTVL(10),DISP(51),ENTEN(51),ENDPT(12),A1(10),A2(10),
                         1 A3(10),A4(10),EMIS(51),YCALC(51),PERROR(51),SD(51),VC(51,51)
                 C
  ISN 0006                COMMON DISP,ENTEN,ENDPT,RO,A1,A2,A3,A4,YCALC,PERROR,EMIS,HEAD,
                         1 WL,A,G,EN,                          SD,VC
  ISN 0007                COMMON NPT,NPNTVL,NTVL
                 C
                 C
  ISN 0008                READ(5,1000)HEAD
  ISN 0009                READ(5,1002)WL,A,G,EN
  ISN 0010                READ(5,1001)NPT,NTVL,ITIME
  ISN 0011                READ(5,1001,ERR=100)(NPNTVL(I),I=1,NTVL)
  ISN 0012                READ(5,1002,ERR=101)(DISP(I),ENTEN(I),SD(I),I=1,NPT)
  ISN 0013                READ(5,1002)ZERO,CAL2,CAL3,CAL4,CAL5,CAL6
  ISN 0014                IF(CAL2.EQ.0.0)CAL2=1.0
  ISN 0016                IF(CAL4.EQ.0.0)CAL4=1.0
  ISN 0018                WRITE(6,2000)HEAD
  ISN 0019                WRITE(6,2001)
  ISN 0020                WRITE(6,2002)NPT,NTVL
  ISN 0021                WRITE(6,2003)(NPNTVL(I),I=1,NTVL)
  ISN 0022                WRITE(6,2005)
  ISN 0023                WRITE(6,2006)(DISP(I),ENTEN(I),SD(I),I=1,NPT)
                 C
                 C
                 C        CONVERT INPUT SCALE READINGS TO INTENSITIES
  ISN 0024                DO 1 I=1,NPT
  ISN 0025              1 DISP(I)=DABS(DISP(I)-CAL3)*CAL4
  ISN 0026                ENDPT(1)=0.0
  ISN 0027                J=1
  ISN 0028                JS=NPNTVL(1)
  ISN 0029                DO 10 I=1,NPT
  ISN 0030                IF(I.NE.JS) GO TO 5
  ISN 0032                J=J+1
  ISN 0033                IF(J.NE.NTVL+1)JS=JS+NPNTVL(J)
  ISN 0035                IF(I.NE.NPT)GO TO 4
  ISN 0037                ENDPT(J)=DISP(NPT)
  ISN 0038                IF(J.EQ.NTVL+1) GO TO 5
  ISN 0040                WRITE(6,3000)HEAD
  ISN 0041                RETURN
  ISN 0042              4 CONTINUE
  ISN 0043                ENDPT(J)=(DISP(I)+DISP(I+1))*0.5
  ISN 0044              5 CONTINUE
  ISN 0045                FAC=SD(I)/ENTEN(I)
  ISN 0046                ENTEN(I)=(ENTEN(I)-ZERO)*CAL2
  ISN 0047                SD(I)=FAC*ENTEN(I)
```

```
ISN 0048         10 CONTINUE
ISN 0049            WRITE(6,2004)(ENDPT(I),I=1,NTVL)
ISN 0050            WRITE(6,2000)HEAD
ISN 0051            WRITE(6,2007)
ISN 0052            WRITE(6,2006)(DISP(I),ENTEN(I),SD(I),I=1,NPT)
ISN 0053            ENDPT(NTVL+1)=DISP(NPT)
ISN 0054            RO=DISP(NPT)
ISN 0055            RETURN
ISN 0056        100 WRITE(6,4000)
ISN 0057       4000 FORMAT('1',1X,'ERROR IN NPTNVL')
ISN 0058        101 WRITE(6,4001)I
ISN 0059       4001 FORMAT(///1X,'I=',I5)
ISN 0060            RETURN
              C
ISN 0061       1000 FORMAT(20A4)
              C
ISN 0062       1001 FORMAT(26I3)
              C
ISN 0063       1002 FORMAT(6E12.0)
              C
ISN 0064       2000 FORMAT('1',20A4)
              C
ISN 0065       2001 FORMAT(1X,'INPUT DATA')
              C
ISN 0066       2002 FORMAT(/1X,'NUMBER OF POINTS=',I3,/1X,'NUMBER OF INTERVALS=',I3)
              C
ISN 0067       2003 FORMAT(1X,'NUMBER OF POINTS PER INTERVAL:',26I3)
              C
ISN 0068       2004 FORMAT(/1X,'BEGINNING POINT OF EACH INTERVAL:',/(1X,1P10E13.6))
              C
ISN 0069       2005 FORMAT(//1X,'INPUT DATA ARRAY')
              C
ISN 0070       2006 FORMAT(1X,3(1PE13.6,8X))
              C
ISN 0071       2007 FORMAT(1X,'INPUT (X,INTENSITY,STD DEV) ARRAY',///)
              C
ISN 0072       3000 FORMAT('1',20A4,/1X,'J DOES NOT MATCH WITH NTVL')
              C
ISN 0073            END
```

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LEVEL 21.7 ( JAN 73 )                     OS/360  FORTRAN H                           DATE  76.271/08.51.10

          COMPILER OPTIONS - NAME=  MAIN,OPT=02,LINECNT=58,SIZE=0000K,
                            SOURCE,EBCDIC,NOLIST,NODECK,LOAD,MAP,NOEDIT,ID,XREF
  ISN 0002              SUBROUTINE INVERT
                  C
  ISN 0003              IMPLICIT REAL*8(A-H,O-Z)
  ISN 0004              INTEGER HEAD(20)
                  C
  ISN 0005              DIMENSION NPNIVL(10),DISP(51),ENTEN(51),ENDPT(12),A1(10),A2(10),
                       1 A3(10),A4(10),EMIS(51),YCALC(51),PERROR(51),SD(51),VC(51,51)
  ISN 0006              DIMENSION YSPLN(10,10),ZSPLN(10,10),WORK(11)
  ISN 0007              COMMON DISP,ENTEN,ENDPT,RO,A1,A2,A3,A4,YCALC,PERROR,EMIS,HEAD,
                       1 WL,A,G,EN,                            SD,VC
  ISN 0008              COMMON NPT,NPNTVL,NTVL
                  C
  ISN 0009              CALL COVCAL
                  C
  ISN 0010              Z1=ENDPT(NTVL)**2
  ISN 0011              Z2=ENDPT(NTVL+1)**2
  ISN 0012              Z12=Z1*Z1
  ISN 0013              Z13=Z12*Z1
  ISN 0014              Z21=Z2*Z2
  ISN 0015              Z22=Z21*Z2
  ISN 0016              C1=A1(NTVL-1)
  ISN 0017              C2=A2(NTVL-1)
  ISN 0018              C3=A3(NTVL-1)
  ISN 0019              C4=A4(NTVL-1)
  ISN 0020              Y1=C1+Z1*(C2+Z1*(C3 +C4*Z1))
  ISN 0021              Y1P= C2+ 2.0* C3* Z1 +3.0* C4*Z12
  ISN 0022              DEN=Z1-Z2
  ISN 0023              C4=(Y1P-2.0*Y1/DEN)/(DEN*DEN)
  ISN 0024              C3=Y1/(DEN*DEN)-C4*(Z1+2.0*Z2)
  ISN 0025              C2=-Z2*(2.0*C3+3.0*C4*Z2)
  ISN 0026              C1=-Z2*(C2+Z2*(C3+Z2*C4))
  ISN 0027              A1(NTVL)=C1
  ISN 0028              A2(NTVL)=C2
  ISN 0029              A3(NTVL)=C3
  ISN 0030              A4(NTVL)=C4
                  C     DISPLAY COEFFICIENTS FOR THE INTENSITY CURVES
  ISN 0031              WRITE(6,1000)HEAD
  ISN 0032         1000 FORMAT('1',20A4)
  ISN 0033              WRITE(6,1001)
  ISN 0034         1001 FORMAT(1X,'CUBIC COEFFICIENTS',///)
  ISN 0035              WRITE(6,1002)
  ISN 0036         1002 FORMAT(2X,'INTERVAL START',2X,'INTERVAL END',7X,'A1',13X,'A2',12X,
                       1'A3',11X,'A4')
  ISN 0037              WRITE(6,1005)(ENDPT(I),ENDPT(I+1),A1(I),A2(I),A3(I),A4(I),
                       2 I=1,NTVL)
  ISN 0038         1005 FORMAT(1P6E15.6)
                  C
                  C     CALCULATE INTENSITIES AND PERCENTAGE ERROR
  ISN 0039              J=1
  ISN 0040              DO 40 I=1,NPT
  ISN 0041              X=DISP(I)**2
  ISN 0042              IF(X.GT.ENDPT(J+1)**2)J=J+1
  ISN 0044              C1=A1(J)
  ISN 0045              C2=A2(J)
```

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```
ISN 0046          C3=A3(J)
ISN 0047          C4=A4(J)
ISN 0048          YCALC(I)=C1+X*(C2 + X*(C3+C4*X))
ISN 0049          PERROR(I)=0.0
ISN 0050          IF(ENTEN(I).EQ.0.0)GO TO 40
ISN 0052          PERROR(I)=100.0*(YCALC(I)-ENTEN(I))/ENTEN(I)
ISN 0053       40 CONTINUE
ISN 0054       41 CONTINUE
ISN 0055          RETURN
ISN 0056          END
```

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LEVEL 21.7 ( JAN 73 )                    OS/360  FORTRAN H                              DATE  76.271/08.51.13

          COMPILER OPTIONS - NAME=  MAIN,OPT=02,LINECNT=58,SIZE=0000K,
                             SOURCE,EBCDIC,NOLIST,NODECK,LOAD,MAP,NOEDIT,ID,XREF
   ISN 0002             SUBROUTINE  EMSCAL(NPT,NTVL,ENDPT,DISP,A1,A2,A3,A4,EMIS)
   ISN 0003             IMPLICIT REAL*8(A-H,O-Z)
   ISN 0004             DIMENSION ENDPT(1),DISP(1),A1(1),A2(1),A3(1),A4(1),EMIS(1)
   ISN 0005             J=1
   ISN 0006             DO 60  I=1,NPT
   ISN 0007             EMIS(I)=0.0
   ISN 0008             X=DISP(I)
   ISN 0009             IF(I.EQ.NPT) GO TO 60
   ISN 0011             IF(X.GT.ENDPT(J+1)) J=J+1
   ISN 0013             EMIS(I)=EMFUN(2.0*A2(J),4.0*A3(J),6.0*A4(J),X,ENDPT(J+1))
   ISN 0014             IF(J.EQ.NTVL) GO TO 60
   ISN 0016             IBOT=J+1
   ISN 0017             DO 50 K=IBOT,NTVL
   ISN 0018             EMIS(I)=EMIS(I)+EMFUN(2.0*A2(K),4.0*A3(K),6.0*A4(K),X,ENDPT(K+1))-
                       2 EMFUN(2.0*A2(K),4.0*A3(K),6.0*A4(K),X,ENDPT(K))
   ISN 0019        50   CONTINUE
   ISN 0020        60   CONTINUE
   ISN 0021             RETURN
   ISN 0022             END
```

```
LEVEL 21.7 ( JAN 73 )                    OS/360  FORTRAN H                                DATE  76.271/08.51.16

          COMPILER OPTIONS - NAME=  MAIN,OPT=02,LINECNT=58,SIZE=0000K,
                         SOURCE,EBCDIC,NOLIST,NODECK,LOAD,MAP,NOEDIT,ID,XREF
  ISN 0002          DOUBLE PRECISION FUNCTION EMFUN(A2,A3,A4,R,Y)
  ISN 0003          IMPLICIT REAL*8(A-H,O-Z)
  ISN 0004          PI=3.14159265
  ISN 0005          F=Y*Y-R*R
  ISN 0006          IF(F.LE.1.0D-30) GO TO 10
  ISN 0008          F=DSQRT(F)
  ISN 0009          T1=A2*F
  ISN 0010          T2=A3*(Y*Y+2.0*R*R)*F/3.0
  ISN 0011          T3=F*(3.0*Y**4+4.0*R*R*Y*Y+8.0*R**4)/15.0
  ISN 0012          T3=A4*T3
  ISN 0013       20 CONTINUE
  ISN 0014          T1=T1+T2+T3
  ISN 0015          EMFUN=-T1/PI
  ISN 0016          RETURN
  ISN 0017       10 EMFUN=0.0
  ISN 0018          RETURN
  ISN 0019          END
```

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LEVEL 21.7 ( JAN 73 )                      OS/360  FORTRAN H                                  DATE  76.271/08.51.20

          COMPILER OPTIONS - NAME=  MAIN,OPT=02,LINECNT=58,SIZE=0000K,
                             SOURCE,EBCDIC,NOLIST,NODECK,LOAD,MAP,NOEDIT,ID,XREF
  ISN 0002               SUBROUTINE COVCAL
                  C
                  C      .....THIS SUBROUTINE PERFORMS THE ERROR ANALYSIS OPERATIONS.....
                  C
  ISN 0003               IMPLICIT REAL*8(A-H,O-Z)
  ISN 0004               INTEGER HEAD(20)
  ISN 0005               DIMENSION NPNTVL(10),DISP(51),ENTEN(51),ENDPT(12),A1(10),A2(10),
                        1 A3(10),A4(10),EMIS(51),YCALC(51),PERROR(51),SD(51),VC(51,51),
                        2 XTX(67,67),XT(67,51),AV(40)
  ISN 0006               COMMON DISP,ENTEN,ENDPT,RO,A1,A2,A3,A4,YCALC,PERROR,EMIS,HEAD,
                        1 WL,A,G,EN,                                 SD,VC
  ISN 0007               COMMON NPT,NPNTVL,NTVL
  ISN 0008               COMMON /REX/ X(67,51)
                  C
  ISN 0009               F1(A,B)=(2./3.14159265)*DSQRT((A*A)-(B*B))
  ISN 0010               F2(A,B)=(4./(3.*3.14159265))*(DSQRT((A*A)-(B*B)))*((A*A)+(2.*B*B))
  ISN 0011               F3(A,B)=(6./(15.*3.14159265))*(DSQRT((A*A)-(B*B)))*((3.*A*A*A*A)+
                        1 (4.*A*A*B*B)+(8.*B*B*B*B))
                  C
  ISN 0012               NTVL4=4*NTVL
  ISN 0013               NTVL41=NTVL4+1
  ISN 0014               NTVL42=NTVL4-2
  ISN 0015               MD=7*NTVL-3
  ISN 0016               NT10=10*NTVL
                  C
  ISN 0017               DO3I=1,MD
  ISN 0018               DO3J=1,NPT
  ISN 0019             3 XT(I,J)=0.0
  ISN 0020               K1=1
  ISN 0021               K2=0
  ISN 0022               K4=K1+3
  ISN 0023               L=1
  ISN 0024               DO5J=1,NPT
  ISN 0025               K=0
  ISN 0026               DO4I=K1,K4
  ISN 0027               IF(K.EQ.0)GOTO200
  ISN 0029               IF(DISP(J).EQ.0.0)GOTO210
  ISN 0031               XT(I,J)=DISP(J)**K
  ISN 0032               GOTO201
  ISN 0033           200 XT(I,J)=1.0
  ISN 0034               GOTO201
  ISN 0035           210 XT(I,J)=0.0
  ISN 0036           201 K=K+2
  ISN 0037             4 CONTINUE
  ISN 0038               K2=K2+1
  ISN 0039               IF(K2.NE.NPNTVL(L))GOTO5
  ISN 0041               K1=K4+1
  ISN 0042               K4=K1+3
  ISN 0043               K2=0
  ISN 0044               L=L+1
  ISN 0045             5 CONTINUE
                  C
  ISN 0046               DO6I=1,MD
  ISN 0047               DO6J=1,MD
```

```
ISN 0048          6 XTX(I,J)=0.0
ISN 0049            M2=4
ISN 0050            K1=1
ISN 0051            K=1
ISN 0052            K3=0
ISN 0053            K2=0
ISN 0054            I=1
ISN 0055            JC=I
ISN 0056            M3=1
ISN 0057            MSV=NPNTVL(K)
ISN 0058            DO9J=1,NT10
ISN 0059            SUM=0.0
ISN 0060            DO7L=K1,MSV
ISN 0061            IF(K2.EQ.0)GOTO202
ISN 0063            IF(DISP(L).EQ.0.0)GOTO7
ISN 0065            SUM=SUM+DISP(L)**K2
ISN 0066            GOTO7
ISN 0067        202 SUM=NPNTVL(K)
ISN 0068            L=MSV
ISN 0069          7 CONTINUE
ISN 0070            K3=K3+1
ISN 0071            XTX(I,JC)=SUM
ISN 0072            XTX(JC,I)=SUM
ISN 0073            IF(K3.EQ.M2)GOTO203
ISN 0075            JC=JC+1
ISN 0076            K2=K2+2
ISN 0077            GOTO9
ISN 0078        203 I=I+1
ISN 0079            JC=I
ISN 0080            K3=0
ISN 0081            M2=M2-1
ISN 0082            IF(M2.EQ.0)GOTO204
ISN 0084            K2=K2-(M2-M3)
ISN 0085            M3=M3+1
ISN 0086            GOTO9
ISN 0087        204 K1=MSV+1
ISN 0088            K=K+1
ISN 0089            MSV=MSV+NPNTVL(K)
ISN 0090            M2=4
ISN 0091            K2=0
ISN 0092            M3=1
ISN 0093          9 CONTINUE
              C
ISN 0094            J=1
ISN 0095            K=2
ISN 0096            DO15I=NTVL41,MD,3
ISN 0097            XTX(I,J)=1./2.
ISN 0098            XTX(J,I)=XTX(I,J)
ISN 0099            XTX(I,J+4)=-XTX(I,J)
ISN 0100            XTX(J+4,I)=XTX(I,J+4)
ISN 0101            XTX(I,J+1)=(1./2.)*ENDPT(K)*ENDPT(K)
ISN 0102            XTX(J+1,I)=XTX(I,J+1)
ISN 0103            XTX(I,J+5)=-XTX(I,J+1)
ISN 0104            XTX(J+5,I)=XTX(I,J+5)
ISN 0105            XTX(I,J+2)=(1./2.)*ENDPT(K)**4
ISN 0106            XTX(J+2,I)=XTX(I,J+2)
```

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ISN 0107          XTX(I,J+6)=-XTX(I,J+2)
ISN 0108          XTX(J+6,I)=XTX(I,J+6)
ISN 0109          XTX(I,J+3)=(1./2.)*ENDPT(K)**6
ISN 0110          XTX(J+3,I)=XTX(I,J+3)
ISN 0111          XTX(I,J+7)=-XTX(I,J+3)
ISN 0112          XTX(J+7,I)=XTX(I,J+7)
ISN 0113          XTX(I+1,J+1)=ENDPT(K)
ISN 0114          XTX(J+1,I+1)=XTX(I+1,J+1)
ISN 0115          XTX(I+1,J+5)=-XTX(I+1,J+1)
ISN 0116          XTX(J+5,I+1)=XTX(I+1,J+5)
ISN 0117          XTX(I+1,J+2)=4.*ENDPT(K)**3
ISN 0118          XTX(J+2,I+1)=XTX(I+1,J+2)
ISN 0119          XTX(I+1,J+6)=-XTX(I+1,J+2)
ISN 0120          XTX(J+6,I+1)=XTX(I+1,J+6)
ISN 0121          XTX(I+1,J+3)=3.*ENDPT(K)**5
ISN 0122          XTX(J+3,I+1)=XTX(I+1,J+3)
ISN 0123          XTX(I+1,J+7)=-XTX(I+1,J+3)
ISN 0124          XTX(J+7,I+1)=XTX(I+1,J+7)
ISN 0125          XTX(I+2,J+1)=1.
ISN 0126          XTX(J+1,I+2)=1.
ISN 0127          XTX(I+2,J+5)=-1.
ISN 0128          XTX(J+5,I+2)=XTX(I+2,J+5)
ISN 0129          XTX(I+2,J+2)=6.*ENDPT(K)*ENDPT(K)
ISN 0130          XTX(J+2,I+2)=XTX(I+2,J+2)
ISN 0131          XTX(I+2,J+6)=-XTX(I+2,J+2)
ISN 0132          XTX(J+6,I+2)=XTX(I+2,J+6)
ISN 0133          XTX(I+2,J+3)=15.*ENDPT(K)**4
ISN 0134          XTX(J+3,I+2)=XTX(I+2,J+3)
ISN 0135          XTX(I+2,J+7)=-XTX(I+2,J+3)
ISN 0136          XTX(J+7,I+2)=XTX(I+2,J+7)
ISN 0137          J=J+4
ISN 0138          K=K+1
ISN 0139       15 CONTINUE
            C
ISN 0140          CALL MATINV(XTX,MD)
            C
ISN 0141          DO13I=1,MD
ISN 0142          DO13J=1,NPT
ISN 0143       13 X(I,J)=0.
ISN 0144          DO30I=1,MD
ISN 0145          DO30J=1,NPT
ISN 0146          DO30K=1,MD
ISN 0147       30 X(I,J)=XTX(I,K)*XT(K,J)+X(I,J)
            C
ISN 0148          DO52I=1,NTVL4
ISN 0149       52 AV(I)=0.0
ISN 0150          DO53I=1,NTVL4
ISN 0151          DO53J=1,NPT
ISN 0152       53 AV(I)=X(I,J)*ENTEN(J)+AV(I)
            C
ISN 0153          JJ=1
ISN 0154          DO54I=1,NTVL
ISN 0155          A1(I)=AV(JJ)
ISN 0156          A2(I)=AV(JJ+1)
ISN 0157          A3(I)=AV(JJ+2)
ISN 0158          A4(I)=AV(JJ+3)
```

```
ISN 0159        54 JJ=JJ+4
           C
ISN 0160           DO31I=1,NPT
ISN 0161           DO31J=1,MD
ISN 0162        31 XTX(I,J)=0.0
           C
ISN 0163           NPT1=NPT-1
ISN 0164           J1=1
ISN 0165           J2=2
ISN 0166           DO19I=1,NPT1
ISN 0167           D=DISP(I)
ISN 0168           IF(D.GT.ENDPT(J1+1))GOTO16
ISN 0170           GOTO17
ISN 0171        16 J1=J1+1
ISN 0172           J2=J2+4
ISN 0173        17 XTX(I,J2)=-F1(ENDPT(J1+1),DISP(I))
ISN 0174           XTX(I,J2+1)=-F2(ENDPT(J1+1),DISP(I))
ISN 0175           XTX(I,J2+2)=-F3(ENDPT(J1+1),DISP(I))
ISN 0176           IF(J2.EQ.NTVL42)GOTO19
ISN 0178           K1=J1+1
ISN 0179           J24=J2+4
ISN 0180           DO18K=J24,NTVL4,4
ISN 0181           XTX(I,K)=F1(ENDPT(K1),DISP(I))-F1(ENDPT(K1+1),DISP(I))
ISN 0182           XTX(I,K+1)=F2(ENDPT(K1),DISP(I))-F2(ENDPT(K1+1),DISP(I))
ISN 0183           XTX(I,K+2)=F3(ENDPT(K1),DISP(I))-F3(ENDPT(K1+1),DISP(I))
ISN 0184        18 K1=K1+1
ISN 0185        19 CONTINUE
           C
ISN 0186           DO20I=1,NPT
ISN 0187           DO20J=1,NPT
ISN 0188        20 XT(I,J)=0.0
ISN 0189           DO21I=1,NPT
ISN 0190           DO21J=1,NPT
ISN 0191           DO21K=1,MD
ISN 0192        21 XT(I,J)=XTX(I,K)*X(K,J)+XT(I,J)
           C
ISN 0193           DO22I=1,NPT
ISN 0194           DO22J=1,NPT
ISN 0195        22 XTX(J,I)=XT(I,J)
           C
ISN 0196           DO25I=1,NPT
ISN 0197           DO25J=1,NPT
ISN 0198        25 X(I,J)=0.0
ISN 0199           DO26I=1,NPT
ISN 0200           DO26J=1,NPT
ISN 0201        26 X(I,J)=XT(I,J)*SD(J)*SD(J)
           C
ISN 0202           DO27I=1,NPT
ISN 0203           DO27J=1,NPT
ISN 0204        27 VC(I,J)=0.0
ISN 0205           DO28I=1,NPT
ISN 0206           DO28J=1,NPT
ISN 0207           DO28K=1,NPT
ISN 0208        28 VC(I,J)=X(I,K)*XTX(K,J)+VC(I,J)
           C
ISN 0209           DO29I=1,NPT
```

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ISN 0210         29 VC(I,I)=DSQRT(VC(I,I))
ISN 0211            RETURN
ISN 0212            END
```

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LEVEL 21.7 ( JAN 73 )                    OS/360  FORTRAN H                                 DATE  76.271/08.51.29

          COMPILER OPTIONS - NAME=  MAIN,OPT=02,LINECNT=58,SIZE=0000K,
                             SOURCE,EBCDIC,NOLIST,NODECK,LOAD,MAP,NOEDIT,ID,XREF
  ISN 0002              SUBROUTINE MATINV     (A, NN)
  ISN 0003              IMPLICIT REAL*8(A-H,O-Z)
  ISN 0004              DIMENSION A(67,67),LOCATE(67,3)
  ISN 0005              DO 1 N=1,NN
  ISN 0006            1 LOCATE(N,3)= 0
  ISN 0007              DO 14 N=1,NN
  ISN 0008              AMAX= 0.0D+0
  ISN 0009              DO 6 I=1,NN
  ISN 0010              IF ( LOCATE(I,3).EQ.0 ) GO TO 2
  ISN 0012              GO TO 6
  ISN 0013            2 DO 5 J=1,NN
  ISN 0014              IF ( LOCATE(J,3).EQ.0 ) GO TO 3
  ISN 0016              GO TO 5
  ISN 0017            3 IF(DABS(A(I,J)).GT.AMAX) GO TO 4
  ISN 0019              GO TO 5
  ISN 0020            4 AMAX=DABS (A(I,J))
  ISN 0021              IROW= I
  ISN 0022              JCOL= J
  ISN 0023            5 CONTINUE
  ISN 0024            6 CONTINUE
  ISN 0025              IF ( AMAX.GT.1.0D-15 ) GO TO 7
  ISN 0027              GO TO 18
  ISN 0028            7 LOCATE(N,1)= IROW
  ISN 0029              LOCATE(N,2)= JCOL
  ISN 0030              LOCATE(JCOL,3)= 1
  ISN 0031              IF ( IROW.NE.JCOL ) GO TO 8
  ISN 0033              GO TO 10
  ISN 0034            8 DO 9 J=1,NN
  ISN 0035              SWAP= A(IROW,J)
  ISN 0036              A(IROW,J)= A(JCOL,J)
  ISN 0037            9 A(JCOL,J)= SWAP
  ISN 0038           10 PIVOT= A(JCOL,JCOL)
  ISN 0039              A(JCOL,JCOL)= 1.0D+0
  ISN 0040              DO 11 J=1,NN
  ISN 0041           11 A(JCOL,J)= A(JCOL,J) / PIVOT
  ISN 0042              DO 14 I=1,NN
  ISN 0043              IF ( I.NE.JCOL ) GO TO 12
  ISN 0045              GO TO 14
  ISN 0046           12 F= A(I,JCOL)
  ISN 0047              A(I,JCOL)= 0.0D+0
  ISN 0048              DO 13 J=1,NN
  ISN 0049           13 A(I,J)= A(I,J) - F*A(JCOL,J)
  ISN 0050           14 CONTINUE
  ISN 0051              DO 17 N=1,NN
  ISN 0052              L= NN-N+1
  ISN 0053              IF ( LOCATE(L,1).NE.LOCATE(L,2) ) GO TO 15
  ISN 0055              GO TO 17
  ISN 0056           15 IROW= LOCATE(L,1)
  ISN 0057              JCOL= LOCATE(L,2)
  ISN 0058              DO 16 K=1,NN
  ISN 0059              SWAP= A(K,IROW)
  ISN 0060              A(K,IROW)= A(K,JCOL)
  ISN 0061              A(K,JCOL)= SWAP
  ISN 0062           16 CONTINUE
```

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ISN 0063         17 CONTINUE
ISN 0064            RETURN
ISN 0065         18 PRINT 1000
ISN 0066            RETURN
ISN 0067       1000 FORMAT (1H , 10X, 15HSINGULAR MATRIX)
ISN 0068            END
```

F88-LEVEL LINKAGE EDITOR OPTIONS SPECIFIED LET,MAP
DEFAULT OPTION(S) USED - SIZE=(100352,12288)

MODULE MAP

CONTROL SECTION NAME	ORIGIN	LENGTH	ENTRY NAME	LOCATION	NAME	LOCATION	NAME	LOCATION	NAME	LOCATION
MAIN	00	702								
INPUT	708	790								
INVERT	E98	476								
EMSCAL	1310	410								
EMFUN	1720	27C								
COVCAL	19A0	1094E								
MATINV	122F0	83A								
IHCLSQRT*	12830	158	DSQRT	12830						
IHCFDXPI*	12C90	140	FDXPI#	12C90						
IHCECOMH*	120E0	F61	IBCOM#	120E0	FDIOCS#	12E9C	INTSWTCH	13D26		
IHCCOMH2*	13D48	650	SEQDASD	140C0						
IHCFCVTH*	143A8	1185	ADCON#	143A8	FCVAOUTP	14452	FCVLOUTP	144E2	FCVZOUTP	1463A
			FCVIOUTP	149EE	FCVEOUTP	14EF0	FCVCOUTP	1510A	INT6SWCH	153F3
IHCEFNTH*	15560	542	ARITH#	15560	ADJSWTCH	158FC				
IHCEFIOS*	15AA8	F28	FIOCS#	15AA8	FIOCSBEP	15AAE				
IHCFIOS2*	169D0	52E								
IHCERRM *	16F00	50C	ERRMON	16F00	IHCERRE	16F18				
IHCUOPT *	174E0	300								
IHCETRCH*	177E0	28E	IHCTRCH	177E0	ERRTRA	177E8				
IHCUATBL*	17A70	638								
$BLANKCOM	180A8	5D20								
REX	1DDC8	6AC8								

ENTRY ADDRESS 00
TOTAL LENGTH 24890

****USERPROG DOES NOT EXIST BUT HAS BEEN ADDED TO DATA SET

NOMENCLATURE

A	Matrix of curve fit coefficients
a_{ki}	Coefficient of $x^{2(k-1)}$ in the k^{th} interval
B	Linear transformation matrix, defined in Eqs. (13) and (14)
C	Vector representing right-hand side of least-squares equations, defined by Eqs. (11), (13), and (19)
E	Matrix of emission coefficients
$[E]_{cv}$	Variance-covariance matrix for emission coefficients
F	Represents matrix product, MW
F_1, F_k, F_n	Functions in developing least-squares equations, defined by Eq. (8)
G	Matrix of powers of x, Eqs. (19), (20), (21)
I	Radiance, generally watts/cm^2/sr, but for illustrative purposes, arbitrary units
i,j,k	Dummy subscripts
M	Matrix of coefficients of the curve-fit coefficients Eqs. (23) and (24)
m	Number of points at which emission coefficient is evaluated

m_k	Number of points in the k^{th} interval
N	Submatrix of B, defined in Eqs. (17) and (18)
n	Number of intervals
P_j	Submatrix of R, defined in Eq. (16)
$P_k(x)$	Polynomial in k^{th} interval, Eq. (5)
p	Number of points
Q_j	Submatrix of N, defined in Eq. (18)
R	Submatrix of B, defined in Eqs. (15) and (16), or outer radius of emission source
r	Radial position, generally cm, but for illustrative purposes, arbitrary units
S_j	Submatrix of G, defined in Eq. (21)
S_k	Sum of squares of residuals, defined in Eq. (6)
W	Matrix product $B^{-1}G$
x	Displacement, generally cm, but for illustrative purposes, arbitrary units
Y	Column vector of radiance; also used in place of I in Appendix A
$[Y]_{cv}$	Variance-covariance matrix for radiance

y	Scalar member of Y, radiance
Z_k	Right side abcissa of k^{th} interval
ε	Emission coefficients, generally watts/cm^3/sr, but for illustrative purposes, arbitrary units
λ	Lagrange's undermined multiplier
μ_I	Vector of mean values of radiances
μ_ε	Vector of mean values of emission coefficients
$\mathcal{E}$	Expected value operator
π	3.14159265
$\phi_{k,1}$	Constraint on polynomial at Z_k
$\phi_{k,2}$	Constraint on first derivative at Z_k
$\phi_{k,3}$	Constraint on second derivative at Z_k